

Multiparticle extension of the higher-spin algebra

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Abstract

Multiparticle extension of a higher-spin algebra l is introduced as the Lie superalgebra associated with the universal enveloping algebra $U(l)$. While conventional higher-spin symmetry does not mix n -particle states with different n , multiparticle symmetries do so. Quotients of multiparticle algebras are considered, that act on the space of n -particle states with $0 \leq n \leq k$ analogous to the space of first k Regge trajectories of String Theory. Original higher-spin algebra is reproduced at $k = 1$. Full multiparticle algebras are conjectured to describe vacuum symmetries of string-like extensions of higher-spin gauge theories. Relation of the multiparticle algebras with $3d$ current operator algebras is described. The central charge parameter, to be related to the parameter \mathcal{N} in AdS/CFT correspondence, enters via the definition of supertrace. Extension to higher p -brane-like symmetries is introduced inductively.

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1 Introduction

Higher-spin (HS) gauge theories describe interactions of massless fields of all spins. First example of full nonlinear HS theory was given in the $4d$ case [1], while its modern formulation was worked out in [2] (see [3] for a review). HS gauge theories involve infinite towers of massless (gauge) fields of higher spins. In this respect HS gauge theory is analogous to String Theory which also describes interactions of excitations of states of all spins. These two classes of theories are however different in several respects.

Known HS gauge theories only involve totally symmetric fields while String Theory contains HS fields of various symmetry types. Field spectra of HS gauge theories are somewhat analogous to the first Regge trajectory of String Theory, though describing only massless (gauge) fields of higher spins. On the other hand, String Theory describes only massive higher spins. Another distinction is that HS theories admit consistent interactions only in a curved background which is $(A)dS_d$ in the most symmetric case, while fully consistent formulation of String Theory is available in a Ricci flat background.

It was anticipated for a long time that HS theory and String Theory should be related and, eventually, String Theory should be understood as some HS theory where masses are generated via spontaneous breakdown of HS symmetries (for example, this was conjectured in [4]). Although this conjecture is supported by the analysis of high-energy limit of string amplitudes [5] and passed some nontrivial checks [6, 7, 8, 9], no satisfactory understanding of this relation beyond the free field sector of the tensionless limit of String Theory [10, 11, 12] was available. An interesting idea of singleton string whose spectrum is represented by multiple tensor products of singletons was put forward in [13, 14]. Somewhat similarly, it was recently conjectured [15] that String Theory should admit an interpretation of a theory of bound states of HS gauge theory. Consideration of this paper agrees with these conjectures, specifying a symmetry underlying a string-like extension of HS gauge theory.

Naively, the difference between the two classes of theories is minor. HS theories are formulated in terms of fields $B(Y|X)$ that depend on space-time coordinates X and auxiliary variables Y^A . The latter, depending on a model, can be either spinors [16, 2] or a pair of vectors [17]. The variables Y^A are noncommutative, obeying commutation relations

$$[Y^A, Y^B] = 2C^{AB} \quad (1.1)$$

with some non-degenerate antisymmetric matrix C^{AB} which is either the charge conjugation matrix with spinor indices A, B or has the form $C^{AB} = \epsilon^{\alpha\beta}\eta^{ab}$ with $A = (a, \alpha)$ where $\alpha = 1, 2$, $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha} \neq 0$, a is the vector index of $o(d-1, 2)$ and η^{ab} is an $o(d-1, 2)$ invariant metric. These oscillators are analogous to a pair of string oscillators, say x_1^n and x_{-1}^m , that satisfy

$$[x_1^n, x_{-1}^m] = \eta^{nm}, \quad (1.2)$$

where η^{nm} is Minkowski metric in d dimensions.

It looks like it is enough to let more species of oscillators $Y^A \rightarrow Y_i^A$ ($i = 1, \dots, r$) be present to get HS theory closer to String Theory picture which should emerge in the $r \rightarrow \infty$ limit. This idea is supported by the analysis of unfolded formulation of free mixed symmetry HS

gauge fields which were shown [18, 19, 20, 21] to be naturally described in terms of differential forms $\omega(Y_i|x)$ of various degrees, valued in appropriate tensor $o(d-1, 2)$ -modules realized by polynomial functions $\omega(Y_i|x)$ of oscillators Y_i^A . However, to go beyond the free field level, it is necessary to find such a non-Abelian algebra hs that fields of the unfolded formulation of the theory fit into hs -modules. In particular, 1-forms among $\omega(Y_i|x)$ should be valued in its adjoint representation. A strong criterium, called admissibility condition [22], requires a HS algebra to admit a unitary module that decomposes into direct sum of unitary modules of the space-time symmetry algebra s , whose pattern matches the list of relativistic fields associated with the list of forms $\omega(Y_i|x)$. For symmetric HS fields this is indeed the case [22, 23] due to Flato-Fronsdal theorem [24] and its higher-dimensional generalization [25] which relates tensor product of a pair of scalar and/or spinor unitary modules of the conformal algebra $o(d-1, 2)$ in $d-1$ dimensions to the towers of massless fields in d dimensions. In this realization, the AdS_d HS algebra is identified with the algebra of endomorphisms of the space of single-particle states of conformal fields in $d-1$ dimension. However, no analogue of this construction appropriate for the description of mixed symmetry fields of general type is available. The problem is most obvious for odd d where conformal scalar and spinor fields are the only unitary propagating conformal fields. This indicates that generalization of HS theory to mixed symmetry fields and, eventually, to String Theory, may require some deviation from the standard constructions of HS theory.

1-forms $\omega(Y|x)$ valued in $s \subset hs$, describe vielbein and connection of the spin two gravitational field. In the usual HS theory of symmetric fields, the corresponding gravitational fields were associated with the (sub)algebra of bilinears of the oscillators

$$T^{AB} = \frac{1}{2}\{Y^A, Y^B\}. \quad (1.3)$$

For instance, in the $4d$ spinor realization where $A, B = 1, \dots, 4$, T^{AB} are generators of the AdS_4 algebra $sp(4) \sim o(3, 2)$. For arbitrary dimension d , generators of $o(d-1, 2)$ were identified [17] with the subalgebra of $sp(2(d+1))$ spanned by those T^{AB} that are invariant under the $sp(2)$ subalgebra rotating indices α of $Y^{a\alpha}$.

However straightforward extension of this construction to any number r of oscillators

$$T^{AB} = \frac{1}{2} \sum_{i=1}^r \{Y_i^A, Y_i^B\} \quad (1.4)$$

does not respect the admissibility condition facing the following problem. A natural framework for unitary hs -modules is provided by tensor products of Fock modules where the oscillators Y_i^A act. Let E be the energy operator among T^{AB} . If the lowest energies for spin s fields were $E(s)$ in the $r=1$ hs -module (recall that in the absence of a free mass parameter, lowest energies in AdS_d are scaled in units of the inverse AdS radius), in the tensor product of r such modules energies will increase like $rE(s)$. Since the lowest energy determines the mass of a particle, if it described a massless symmetric field in the $r=1$ case, it will correspond to certain massive (and hence non-gauge) field at higher r . In particular, spin one and two fields become massive at $r > 1$, *i.e.*, the resulting theory can contain neither Yang-Mills theory, nor gravity.

So far, no non-Abelian HS algebra appropriate for description of general mixed symmetry fields was available, though some particular mixed symmetry fields result from gauging of the HS algebra associated with the tensor product of fermions in any dimension [25] as well as with $4d$ conformal HS algebras of [26, 27] and their further $4d$ [28, 29, 30] (for the respective Flato-Fronsdal like theorem see [31]) and higher-dimensional [32, 33] extensions.

A well-known feature of conventional formulation of String Theory, which seems to be closely related to the above discussion of HS theory, is that its consistent generalization to AdS background is far from being trivial. Indeed, as in any relativistic theory, Lorentz symmetry acts on all space-time (spinor-)tensors both in HS theory and in String Theory. Hence, Lorentz generators should have a form (1.4) where summation is over all modes that carry space-time indices. That commutator of translations (transvections) in AdS algebra gives Lorentz generators requires the AdS translation generators to be built from all modes. However, as in HS theory, this would immediately lead to wrong (infinite) vacuum energy of graviton. Hence, translation generators in String Theory are built solely from zero modes, which construction admits no AdS deformation (see however interesting work [34] where an extended formulation of String Theory, that avoids this problem, was proposed). This feature of String Theory indicates that the straightforward construction via tensoring of oscillators is too naive in the both cases.

In this paper, we propose a class of algebras that extend usual HS algebras in a String Theory fashion, avoiding the most obvious problems mentioned above. The proposed construction was deduced from the analysis of [35] of current operator algebra of $3d$ massless free theory, which generates symmetries of the space of multiparticle states of the AdS_4 HS theory and its boundary image. Hence, we call them multiparticle algebras. The purely algebraic approach proposed in this paper provides an efficient tool for the description of the current operator algebra of [35], leading to manifest formulae for the current OPE in Section 4. One of the surprising outputs of this construction is that OPE's of currents with different number \mathcal{N} of constituent free fields are described by different basis choices in the same multiparticle algebra. The respective bases are uniquely fixed by the conditions that (i) Wick theorem, is respected as the characteristic property of free field theory and (ii) the central term properly depends on \mathcal{N} . Since different free theories are described by the same multiparticle algebra, it is appealing to speculate that, using the same multiparticle algebra, it may even be possible to formulate nonlinear conformal systems not respecting Wick theorem (see also Conclusion).

It should be stressed that our construction applies to a very general class of theories including the $4d$ $N = 4$ SYM boundary theory closely related to conventional Superstring Theory. Similarly, the analysis of current operator algebra of [35] goes beyond the $3d$ case, allowing in particular to evaluate n -point functions of $4d$ conformal currents.

The formal definition is simple. Let a HS algebra $h_u(V)$ be the Lie algebra of maximal symmetries of V , *i.e.*, of the free theory of fields Φ that have V as the space of single-particle states. Multiparticle algebras $m_u(V)$ are appropriate real forms of the complex Lie superalgebras associated with the universal enveloping algebras $U(h_u(V))$. Multiparticle algebra acts on the space of all multiparticle states in a theory where V is the space of single-particle states.

Algebras $m_u^k(V)$ act on the space of r -particle states with $r \leq k$. $h_u(V) \subset m_u^k(V)$ for

any $k \geq 1$. In particular, $m_u^1(V) = h_u(V) \oplus u(1)$ where $u(1)$ represents the symmetry of the physical vacuum which is the space of 0-particle states. Algebras $m_u^k(V)$ are certain quotients of $m_u(V)$ and should be associated with a theory which roughly speaking describes first k Regge trajectories of String Theory. $m_u(V)$ should be associated with the full-fledged string-like extension of HS theories. We believe that the proposed scheme has a potential to unify String Theory and HS theory within a theory which contains both of them as different particular cases and/or limits.

The paper is organized as follows. In Section 2 general structure of known HS algebras is recalled. In Section 3 we present construction of associative multiparticle algebra which is illustrated in Section 3.6 by the example of Weyl algebra. It is applied to description of current operator algebras of [35] in Section 4 and to extension of every HS algebra $h_u(V)$ to multiparticle algebras $m_u(V)$ and $m_u^k(V)$ in Section 5, where further generalizations of multiparticle algebras to be associated with p -brane extensions of String Theory are also introduced. In Conclusion, some properties of yet hypothetical string-like HS theory are briefly discussed as well as possible extension of the obtained results to non-free current operator algebras.

2 Higher-spin algebras

From the perspective of bulk HS gauge theories in AdS , HS algebras represent global symmetry of a maximally symmetric vacuum solution of the nonlinear HS gauge theory in question. They should be distinguished from local HS symmetries of HS gauge theories, resulting from gauging (localization) of global HS symmetries along with their further field-dependent deformation.¹ In this paper, we focus on the global multiparticle algebras, which provide starting point for the search of the full-fledged nonlinear multiparticle gauge theories.

All HS algebras underlying known nonlinear HS gauge theories admit the following realization. Let V_Φ be the space of single-particle states of a set of free unitary conformal fields Φ . We will use notation $H(V_\Phi)$ for the complex associative algebra of endomorphisms $End(V_\Phi; \mathbb{C})$. As such, $H(V_\Phi)$ is closely related to the algebra of all symmetries of the free field theory of Φ^i as is most easily seen from the unfolded dynamics approach (see e.g. [41]).

For an algebra A with the product law \star (from now on by algebra we mean associative algebra if not specified otherwise), $l(A)$ denotes the associated Lie (super)algebra with the (graded) commutator

$$[a, b]_\star = a \star b \mp b \star a \quad \forall a, b \in A \quad (2.1)$$

as a Lie product. Then the HS algebra $h_u(V_\Phi)$ is the real form of $l(H(V_\Phi))$ singled out by the conditions

$$\sigma(a) = a \quad (2.2)$$

with such conjugation σ of $l(H(V_\Phi))$ ($\sigma(ia) = -i\sigma(a)$, $\sigma^2 = Id$) that the corresponding sym-

¹The latter phenomenon is typical for any theory of gravity where diffeomorphisms can be interpreted as a deformation of localized Poincaré or $(A)dS$ transformations by curvature-dependent terms [36, 37, 38, 39, 40].

metry transformations of V_Φ are unitary.² Given conformal fields Φ in d dimensions, $h_u(V_\Phi)$ can be interpreted either as conformal HS algebra in d dimensions or as AdS_{d+1} HS algebra.

$hu(V_\Phi)$ admits further truncations of the orthogonal and symplectic types induced by an involutive antiautomorphism ρ of $H(V_\Phi)$

$$\rho(ab) = \rho(b)\rho(a), \quad \rho^2 = Id. \quad (2.3)$$

For Φ^i carrying color index $i = 1, 2, \dots, n$, there are two options for the extension of ρ that lead to two types of HS algebras. Namely, if ρ was an antiautomorphism of the model with a single field Φ , its color extension ρ^{col} is

$$\rho^{col}(\Phi^i) = \eta_{ij}\rho(\Phi^j), \quad (2.4)$$

where η_{ij} is some nondegenerate matrix. (Note that ρ^{col} maps left and right $H(V_\Phi)$ -modules to each other.) The truncation condition is

$$\rho^{col}(a) = -i^{p(a)}a, \quad (2.5)$$

where $p(a) = 0$ or 1 is the boson-fermion parity of a . Depending on whether η_{ij} is symmetric or antisymmetric, this gives the algebras $h_o(V_\Phi)$ or $h_{usp}(V_\Phi)$, respectively. In the case of a single field Φ ($n = 1$), $h_o(V_\Phi)$ is the minimal HS algebra. For HS algebras associated with symmetric fields of integer spins, $h_o(V_\Phi)$ contains even spins. Note that if η_{ij} has no definite symmetry, the subalgebra singled out by (2.5) is a direct sum of algebras $h_o(V_\Phi)$ and $h_{usp}(V_\Phi)$ with smaller n . (For more detail we refer the reader to [16, 23] where this construction was originally applied to AdS_4 HS algebras.)

HS algebras available in the literature belong to the three classes $h_u(n_{s_1}, n_{s_2}, \dots | d)$, $h_o(n_{s_1}, n_{s_2}, \dots | d)$ and $h_{usp}(n_{s_1}, n_{s_2}, \dots | d)$ with $0 \leq s_1 < s_2 < s_3 \dots$. Here d is dimension of space-time where a set of conformal fields Φ^i contains n_{s_1} fields of spin s_1 , n_{s_2} fields of spin s_2 , etc. $h_u(n_{s_1}, n_{s_2}, \dots | d)$ is the algebra $hu(V_\Phi)$ for the corresponding set of fields Φ^i while $h_o(n_{s_1}, n_{s_2}, \dots | d)$ and $h_{usp}(n_{s_1}, n_{s_2}, \dots | d)$ are its subalgebras singled out by the condition (2.4). In principle, one can also consider the case where different fields Φ^i live in space-times of different dimensions. Though no algebras of this type were so far considered in the literature, recent results of [42], where it was shown that current interactions of $4d$ massless fields acquire natural interpretation in terms of a mixed system of $4d$ and $6d$ conformal fields, suggest that they may also be of interest. For such algebras one can use notation $h_{\dots}(n_{d_1 s_1}, n_{d_2 s_2}, \dots)$.

The list of ghost-free propagating conformal fields in d dimensions depends on whether d is even or odd. As shown in [43, 44], apart from massless scalar and spinor in any dimension, only mixed symmetry fields with field strengths described by rectangular Young diagrams of height $d/2$ in even space-time dimension correspond to unitary theories. Hence, for even d , conformal massless fields are characterized by a single spin parameter s associated with a length s or $s - \frac{1}{2}$ of tensor or spinor-tensor Young diagram, respectively. Conformal scalar and spinor correspond to Young diagrams of zero length, hence making sense for odd d as well.

²Note that the antihermiticity condition implying unitarity of symmetry transformations requires the conjugation σ of $l(H_\Phi)$ be generated by an involution \dagger of H_Φ via $\sigma(a) := -a^\dagger$ in the Lie algebra (*i.e.*, bosonic) case. Since involution reverses the order of product factors, $(ab)^\dagger = b^\dagger a^\dagger$, the antihermiticity condition does not allow us to start with a real algebra H_Φ (anti-Hermitian operators do not form an associative algebra).

The presented construction of HS algebras is closely related to Flato-Fronsdal-type theorems on the relation between tensor product of conformal fields in d dimensions and massless fields in AdS_{d+1} . Indeed, HS gauge fields associated with the AdS_{d+1} HS algebra are valued in the algebra of operators that act in V_Φ , which is $V_\Phi^* \otimes V_\Phi$ as a linear space. An important feature of HS gauge theories is [16, 17] that Weyl 0-forms, which contain all degrees of freedom of the AdS_{d+1} system, are valued in the so-called twisted adjoint module which is isomorphic to $V_\Phi^* \otimes V_\Phi$ as a linear space. This implies that degrees of freedom of the AdS_{d+1} HS theory belong to the module equivalent to the tensor square of the conformal module V_Φ in d dimensions (here we do not distinguish between vector spaces V_Φ^* and V_Φ which are isomorphic in the unitary case). Hence, the construction of HS algebras is such that the spectrum of fields of the bulk HS theory is designed to result from the tensor product of boundary conformal fields. This is in kinematical agreement with the idea of AdS/CFT correspondence because $V_\Phi \otimes V_\Phi$ is the space of conformal conserved currents of the theory of free fields Φ^i as is most directly seen in the unfolded dynamics approach [45], which fact is of course not surprising given that $V_\Phi \otimes V_\Phi$ is the space of conformal HS symmetries. Hence HS algebras $h_{\dots}(n_{s_1}, n_{s_2}, \dots | d)$ are properly designed to support AdS/CFT correspondence between boundary conformal theories and bulk HS theories, the issue which acquired a lot of interest during recent years. (See, e.g., [46, 47, 48, 28, 49, 50, 51, 52, 53, 54, 55, 56, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 15, 79, 80]. For reviews and more references see also [81, 82, 83].) Of course, being applicable to free fields, the above consideration has to be reanalyzed at the interaction level. For example, as shown in [74], except for two particular reductions of the HS gauge theory which, in accordance with the theorem of Maldacena and Zhiboedov [72], are dual to the boundary theory of free currents, all other nonlinear AdS_4 HS gauge theories turn out to be dual to a $3d$ conformal HS gauge theory of interacting currents.

Although original AdS_4 HS algebras were obtained in [84, 85, 86] from different arguments [87] aimed at reformulation of HS theory in terms of differential forms, that eventually led to its unfolded formulation [16, 1, 2], they belong to the class of HS algebras discussed above. Even with no reference to symmetries of $3d$ unitary conformal fields, original AdS_4 HS algebras were interpreted as algebras of $3d$ conformal HS gauge theory by Fradkin and Linetsky in [91].

Specifically, the original bosonic AdS_4 HS algebra found in [84] is $h_u(1_0|3)$. Its extension to $h_u(1_0, 1_{1/2}|3)$ was proposed in [86] and to $h_{\dots}(n_0, m_{1/2}|3)$ in [23] where they were called $h_{\dots}(n, m|4)$. Other way around, $4d$ conformal HS algebras introduced by Fradkin and Linetsky in [27] in the context of $4d$ conformal HS gauge theory where later interpreted as AdS_5 HS algebras in [88, 89, 29, 30]. All these algebras are $h_{\dots}(n_0, n_{1/2}, n_1|4)$ where $n_0, n_{1/2}$ and n_1 are numbers of fields of respective spins in a supermultiplet of $4d$ N -extended conformal superalgebra with $N = 1, 2, 4$ including the case of $N = 4$ SYM multiplet $h_u(n_0, n_{1/2}, n_1|4)$ with $n_0 = 6n_1$, $n_{1/2} = 4n_1$. Their realization in terms of $4d$ boundary fields was considered in [28] including the generalization to $h_{\dots}(n_{s_1}, n_{s_2}, \dots |4)$ with nonzero n_s at $s > 1$. Extension to $h_{\dots}(n_s|d)$ with any even $d \geq 4$ was given in [33]. Algebra $h_o(1_0|6)$ interpreted as the minimal AdS_7 HS algebra was considered in [32]. In [28], algebras $h_{\dots}(1_0, 1_1, 1_2 \dots \infty |4)$ and $h_{\dots}(1_0, 1_{1/2}, 1_1 \dots \infty |4)$ (the case of $M = 4$ in notations of [28]) and $h_{\dots}(1_0, 1_1, 1_2 \dots \infty |6)$ and $h_{\dots}(1_0, 1_{1/2}, 1_1 \dots \infty |6)$ (the case of $M = 8$ in notations of [28]) were identified with $l(A_M)$ where

Weyl algebra A_M is the algebra of various polynomials of M pairs of oscillators. $h_u(1_0|d)$ was identified as the algebra of conformal HS symmetries of a massless scalar by Eastwood in [90] and was used for the construction of HS gauge theories in AdS_{d+1} in [17] where it was also extended to $h_u(n_0|d)$. HS superalgebras $h_u(n_0, m_{1/2}|d)$ were introduced in [25].

All HS algebras listed above admit realizations in terms of Weyl algebras, which are particularly useful for the formulation of nonlinear HS gauge theories of [2, 17]. There are two types of constructions mentioned in Introduction. The one with elementary spinor oscillators was used in [85, 86, 91, 26, 23, 88, 89, 28, 29, 30]. That with elementary oscillators carrying vector indices used in [17, 25, 33] applies to HS models in any dimension. Being closely related to twistor theory, the spinor realization is likely to be both simpler and deeper.

For example, HS algebras $h_u(n_0, m_{1/2}|3)$ were shown in [23] (where they were denoted $hu(n, m|4)$) to be realized by matrices

$$P_i^j(Y) = \begin{array}{c} n \\ m \end{array} \begin{array}{|c|c|} \hline & \begin{array}{c} n \\ m \end{array} \\ \hline \begin{array}{c} n \\ m \end{array} & \begin{array}{c} n \\ m \end{array} \\ \hline \end{array} \begin{array}{cc} P^E_{i'j'}(Y) & P^O_{i'j''}(Y) \\ P^O_{i''j'}(Y) & P^E_{i''j''}(Y) \end{array} \quad (2.6)$$

where matrix valued polynomials $P^E(Y)$ and $P^O(Y)$ are respectively even and odd functions of the oscillators Y_A ($A = 1, 2, 3, 4$ is the $4d$ Majorana spinor index) that obey the star-product commutation relations (1.1) where C_{AB} is the $4d$ charge conjugation matrix.

The space V_Φ of single-particle states of n massless scalars and m massless spinors in three dimensions is realized as the direct sum of n even subspaces F_0 and m odd subspaces F_1 of the Fock module F

$$F : \quad f(Y_+^a)|0\rangle \quad Y_-^a|0\rangle = 0, \quad (2.7)$$

where Y_\pm^a is a pair of mutually conjugated canonical oscillators in the set Y^A . F_0 and F_1 are spanned, respectively, by even and odd functions $f(Y_+^a)$.

That HS algebras are naturally realized in terms of Weyl algebra is not accidental. As algebra of endomorphisms of a single-particle space V_Φ of conformal fields, the HS algebra can be represented by differential operators of various degrees acting in V_Φ . The latter belong to the Weyl algebra which is the algebra of various differential operators with polynomial coefficients. Then V_Φ is represented as a Fock module of the star-product algebra or some its quotient.

The list of full nonlinear HS gauge theories with propagating HS gauge fields known so far, that admit HS algebras as algebras of symmetries of their maximally symmetric vacua, is much shorter than the list of HS algebras given above. It includes AdS_4 theories based on $h_{\dots}(n_0, m_{1/2}|3)$ [1, 2] and AdS_{d+1} theories based on $h_{\dots}(n_0|d+1)$ [17]. (For HS theories in AdS_{d+1} with $d \leq 2$ not considered in this paper, where HS gauge fields carry no degrees of freedom, see [3] and references therein.) The problem of construction of full nonlinear HS theories associated with other HS algebras remains open, though some partial results on the construction of cubic interactions in the respective theories were obtained in [89, 30, 92].

Although the construction of HS algebras sketched above can be used for description of particular mixed symmetry fields (see also [31]), it can unlikely be applied to the generic case rich enough to incorporate String Theory. Hence, some strategy change is needed. Before going into technical detail in the next section, we comment on the general idea.

In the literature (see, e.g., [20, 31, 93]), the construction of HS algebras is often related to the universal enveloping algebra $U(s)$ of the space-time (conformal) symmetry (super)algebra s . If the s -module V_Φ is irreducible, $H(V_\Phi)$ is isomorphic to $U(s)/I_{V_\Phi}$ where I_{V_Φ} is the ideal of $U(s)$ which consists of those its elements that annihilate V_Φ . This is tautologically the case for a single field Φ since this just means that its single-particle states form an irreducible s -module. However, identification of HS algebras with quotients of $U(s)$ may be misleading because the algebras $H(V_\Phi)$ differ from $U(s)/I_{V_\Phi}$ for reducible V_Φ . Indeed, since $H(V_\Phi)$ is the maximal algebra acting on V_Φ ,

$$U(s)/I_{V_\Phi} \subset H(V_\Phi). \quad (2.8)$$

Isomorphism $H(n_0, n_{1/2}, \dots) \sim U(s)/I_{V_\Phi}$ takes place only for irreducible V_Φ , *i.e.*, iff $n_{s_0} = 1$ and $n_s = 0$ at $s \neq s_0$ for some s_0 .

Given algebra A , we introduce associative multiparticle algebra $M(A)$ as $U(l(A))$. The algebra $H(V_\Phi)$ of endomorphisms of V_Φ , which underlies the construction of HS algebra $h_u(V_\Phi)$, gives rise to $M(H(V_\Phi))$. Being defined as universal enveloping of $h_u(V_\Phi)$, $M(H(V_\Phi))$ acts on every $h_u(V_\Phi)$ -module. In particular, it acts on the space

$$\mathcal{V}_\Phi = \sum_{n=0}^{\infty} \oplus V_\Phi^n, \quad V_\Phi^n = \text{Sym} \underbrace{V_\Phi \otimes \dots \otimes V_\Phi}_n \quad (2.9)$$

which is nothing else as the space of all multiparticle states of the fields Φ . Multiparticle algebra associated with the fields Φ will be identified with the appropriate real form of the Lie (super)algebra $l(M(H(V_\Phi)))$ or some of its quotients considered in Section 3.

As discussed in Section 3.5, apart from the simplest possibility where $l(M(H(V_\Phi)))$ acts independently on every V_Φ^n , multiparticle algebras admit representations mixing V_Φ^n with different n . Relating multiparticle states of the field theory of Φ such as, e.g., $N = 4$ SYM theory, multiparticle algebras look particularly appealing in the String Theory context. The problem of increase of lowest energies discussed in Introduction is avoided because a multiparticle algebra contains $h_u(V_\Phi)$ as subalgebra that acts on $V_\Phi \in \mathcal{V}_\Phi$ as in the original HS theory, hence having the same weights (in particular, energies) in this sector. The same time, that the multiparticle algebra is much smaller than the maximal symmetry algebra $h_u(\mathcal{V}_\Phi)$ acting on the space of all multiparticle states of Φ should leave enough flexibility for description of interacting fields Φ .

3 Associative multiparticle algebra

3.1 Definition

Let A be an algebra with the product law \star and basis elements t_i obeying

$$t_i \star t_j = f_{ij}^k t_k. \quad (3.1)$$

Associativity implies

$$f_{ij}^k f_{kl}^n = f_{ik}^n f_{jl}^k. \quad (3.2)$$

In HS algebras, t_i denotes the infinite set of elements $t_i = (1, Y^A, Y^A Y^B, \dots)$ while \star is the star product on functions of Y .

Algebra $M(A)$ is defined as follows. As a linear space, it is isomorphic to direct sum of all symmetric tensor degrees of A

$$M(A) = \sum_{n=0}^{\infty} \oplus \text{Sym} \underbrace{A \otimes \dots \otimes A}_n. \quad (3.3)$$

A natural basis of $M(A)$ is provided by symmetrized tensor product monomials

$$T_{i_1 \dots i_n} = \text{Sym} t_{i_1} \otimes \dots \otimes t_{i_n}, \quad T_{\dots j \dots k \dots} = T_{\dots k \dots j \dots} \quad \forall j, k. \quad (3.4)$$

Let A^* be the space of linear functionals on A , i.e.,

$$\alpha \in A^* : \quad \alpha = \sum_i \alpha_i t^{*i}, \quad t^{*i}(t_j) = \delta_j^i, \quad (3.5)$$

where $\{t^{*i}\}$ is the basis of A^* dual to $\{t_i\}$. $M(A)$ is the algebra of functions $F(\alpha)$ on A^* with the product law

$$F(\alpha) \circ G(\alpha) = F(\alpha) \exp \left(\overleftarrow{\partial} \frac{f_{ij}^n \alpha_n}{\partial \alpha_i} \overrightarrow{\partial} \right) G(\alpha), \quad (3.6)$$

where derivatives $\overleftarrow{\partial}$ and $\overrightarrow{\partial}$ act on F and G , respectively. An elementary computation gives

$$((F_1 \circ F_2) \circ F_3)(\alpha) = \exp \left(f_{ij}^n \alpha_n \sum_{\gamma < \beta=1,2,3} \frac{\partial^2}{\partial \alpha_{\gamma i} \partial \alpha_{\beta j}} + f_{ij}^n f_{nm}^k \alpha_k \frac{\partial^3}{\partial \alpha_{1i} \partial \alpha_{2j} \partial \alpha_{3m}} \right) F_1(\alpha) F_2(\alpha) F_3(\alpha), \quad (3.7)$$

$$(F_1 \circ (F_2 \circ F_3))(\alpha) = \exp \left(f_{ij}^n \alpha_n \sum_{\gamma < \beta=1,2,3} \frac{\partial^2}{\partial \alpha_{\gamma i} \partial \alpha_{\beta j}} + f_{in}^k f_{jm}^n \alpha_k \frac{\partial^3}{\partial \alpha_{1i} \partial \alpha_{2j} \partial \alpha_{3m}} \right) F_1(\alpha) F_2(\alpha) F_3(\alpha), \quad (3.8)$$

where $\frac{\partial}{\partial \alpha_{\beta i}}$ acts on $F_{\beta}(\alpha)$ ($\beta = 1, 2, 3$). Hence, the A -associativity (3.2) implies associativity of the product \circ of $M(A)$.³ Note that $A \subset M(A)$ is represented by linear functions on A^* . Hence, \star product acts on linear functions on A^* according to

$$\alpha_j \star \alpha_k = f_{jk}^m \alpha_m. \quad (3.9)$$

Note however that $A \circ A$ does not belong to A .

³Note that somewhat similar (though different in some important details) algebras of oscillators were mentioned in [13] in the context of singleton strings aimed at the description of multisingleton states which is another name for multiparticle states.

Algebra $M(A)$ is unital, with the unit element Id identified with $F(\alpha) = 1$. Hence

$$M(A) = \mathbb{K} \oplus M'(A), \quad (3.10)$$

where \mathbb{K} is the field over which A and $M(A)$ were defined. (In HS context, the most important case is $\mathbb{K} = \mathbb{C}$.) Indeed, from (3.6) it follows that unit element never appears on the *r.h.s.* of $F \circ G$ if $F(\alpha)$ and/or $G(\alpha)$ is a homogeneous monomial of non-zero degree.

The \mathbb{Z}_2 grading of $M(A)$ is induced by that of A

$$F((-1)^{\pi(\alpha)}\alpha) = (-1)^{\pi(F)}F(\alpha). \quad (3.11)$$

$M(A)$ is isomorphic to the universal enveloping algebra $U(l(A))$

$$M(A) \sim U(l(A)). \quad (3.12)$$

Indeed, by definition of $l(A)$,

$$[t_i, t_j]_\star = g_{ij}^k t_k, \quad g_{ij}^k = f_{ij}^k - f_{ji}^k. \quad (3.13)$$

On the other hand, from (3.6) it follows that

$$\alpha_i \circ \alpha_j - \alpha_j \circ \alpha_i = g_{ij}^k \alpha_k. \quad (3.14)$$

Along with associativity of $M(A)$, Eq. (3.10) and the fact that $M(A)$ is isomorphic to $U(l(A))$ as a linear space, Eq. (3.14) proves (3.12). Concise form of the product law (3.6) is specific to the case where a Lie algebra l of $U(l)$ is associated with an associative algebra A , *i.e.*, $l = l(A)$.

The following useful property of $M(A)$ is a simple consequence of Eq. (3.6)

$$\forall f, g \in A: \quad \exp f(\alpha) \circ \exp g(\alpha) = \exp(f \bullet g)(\alpha), \quad (3.15)$$

where

$$f \bullet g := f + g + f \star g = (f + e_\star) \star (g + e_\star) - e_\star \in A \quad (3.16)$$

(recall that $f, g \in A$ implies that $f(\alpha)$ and $g(\alpha)$ are linear in α). Associativity of \star implies associativity of \bullet

$$(f \bullet g) \bullet h = f \bullet (g \bullet h) = (f + e_\star) \star (g + e_\star) \star (h + e_\star) - e_\star. \quad (3.17)$$

Here and below e_\star denotes the unit element of A . Note however that the product \bullet is associative even if A is not unital.

Let

$$G_\nu = \exp(\nu) \in M(A), \quad \nu = \nu^i \alpha_i, \quad (3.18)$$

where $\nu^i \in \mathbb{K}$ are free parameters. Eq.(3.15) gives

$$G_\nu \circ G_\mu = G_{\nu \bullet \mu}. \quad (3.19)$$

This formula is convenient for practical computations with G_ν used as the generating function for elements of $M(A)$ resulting from differentiation over ν^i .

3.2 Linear maps

Algebra $M(A)$ is double filtered in the following sense. Let V_n be the linear space of order n polynomials of α_i . From Eqs. (3.6), (3.14) it follows that for any $F_n \in V_n$ and $F_m \in V_m$

$$F_n \circ F_m \in V_{n+m}, \quad F_n \circ F_m - F_m \circ F_n \in V_{n+m-1}. \quad (3.20)$$

This property holds for any universal enveloping algebra (see, e.g., [94]).

A linear map of $M(A)$ to itself is represented by

$$\mathcal{U}(\alpha, a) = \sum_{m,n=0}^{\infty} \mathcal{U}^{i_1 \dots i_m}_{j_1 \dots j_n} \alpha_{i_1} \dots \alpha_{i_m} a^{j_1} \dots a^{j_n} \quad (3.21)$$

with

$$\mathcal{U}(\alpha, a)[F] = \mathcal{U}(\alpha, \frac{\vec{\partial}}{\partial \alpha}) F(\alpha), \quad (3.22)$$

where derivatives $\frac{\vec{\partial}}{\partial \alpha}$ act on $F(\alpha)$. This formula can be interpreted as representing the action of the normal-ordered oscillator algebra with the generating elements α_i and a^j acting on the Fock module spanned by $F(\alpha)|0\rangle$ with $a^i|0\rangle = 0$. To respect the double filtration property, mapping order- n polynomials to order- n polynomials, $\mathcal{U}(\alpha, a)$ should obey

$$\mathcal{U}^{i_1 \dots i_m}_{j_1 \dots j_n} = 0 \quad \text{at} \quad m > n. \quad (3.23)$$

Maps of this class, which we call filtered, are of most interest in this paper.

In these terms, the unit map is $\mathbf{Id} = 1$. The map induced by a linear map $u(t_i) = u_i^j t_j$ of A is represented by

$$\mathcal{U}(\alpha, a) = \exp(\alpha_i u_j^i a^j - \alpha_i a^i). \quad (3.24)$$

Consider maps of the form

$$U(f) \equiv \mathcal{U}(\alpha, a|f) = \phi \exp(\alpha_i f^i(a)) \quad (3.25)$$

with some α -independent coefficients $f^i(a)$ and constant ϕ . The map $U(f)$ is filtered provided that $f^i(a)$ is at least linear in a , i.e.,

$$f^i(0) = 0. \quad (3.26)$$

Interpreting a as parameters, we can identify any $f(\alpha) = \sum_i f^i \alpha_i \in M(A)$ with $f(t) = \sum_i f^i t_i \in A$. For $U(f)$ (3.25) acting on G_ν (3.18) we obtain

$$U(f)(G_\nu) = \exp(\tilde{f}^i(\nu) \alpha_i), \quad \tilde{f}^i(\nu) = \nu^i + f^i(\nu). \quad (3.27)$$

Hence, Eq. (3.15) gives

$$U(f)(G_\nu) \circ U(g)(G_\mu) = \exp((\tilde{f}(\nu) \bullet \tilde{g}(\mu))(\alpha)), \quad (3.28)$$

where $\tilde{f}(\nu)$ and $\tilde{g}(\mu)$ are now interpreted as elements of A , i.e., $\tilde{f}(\nu) = \tilde{f}^i(\nu) \alpha_i$.

Important classes of linear maps U of $M(A)$ onto itself are represented by automorphisms

$$\mathcal{T}(G_1 \circ G_2) = \mathcal{T}(G_1) \circ \mathcal{T}(G_2) \quad (3.29)$$

and antiautomorphisms

$$\mathcal{R}(G_1 \circ G_2) = \mathcal{R}(G_2) \circ \mathcal{R}(G_1) \quad (3.30)$$

for $\forall G_{1,2} \in M(A)$. To see whether or not \mathcal{T} and \mathcal{R} are, respectively, automorphism and antiautomorphism of $M(A)$, it is enough to check these properties for $G_1 = G_\nu$ and $G_2 = G_\mu$ with arbitrary ν and μ , hence solving the equations

$$\mathcal{T}(G_\nu) \circ \mathcal{T}(G_\mu) = \mathcal{T}(G_{\nu \bullet \mu}), \quad (3.31)$$

$$\mathcal{R}(G_\mu) \circ \mathcal{R}(G_\nu) = \mathcal{R}(G_{\nu \bullet \mu}). \quad (3.32)$$

Let τ and ρ be, respectively, an automorphism and antiautomorphism of A , *i.e.*,

$$\tau(a \star b) = \tau(a) \star \tau(b), \quad \rho(a \star b) = \rho(b) \star \rho(a) \quad \forall a, b \in A. \quad (3.33)$$

In terms of basis elements t_i and structure coefficients f_{ij}^k this means that matrices τ_i^j and ρ_i^j defined via

$$\tau(t_i) = \tau_i^j t_j, \quad \rho(t_i) = \rho_i^j t_j \quad (3.34)$$

obey

$$\tau_i^{i'} \tau_j^{j'} f_{i'j'}^k = f_{ij}^{k'} \tau_{k'}^k, \quad \rho_i^{i'} \rho_j^{j'} f_{i'j'}^k = f_{ji}^{k'} \rho_{k'}^k. \quad (3.35)$$

Eq. (3.6) implies that τ and ρ induce automorphism \mathcal{T} and antiautomorphism \mathcal{R} of $M(A)$

$$\mathcal{T}(F(\alpha)) = F(\tau(\alpha)), \quad \mathcal{R}(F(\alpha)) = F(\rho(\alpha)). \quad (3.36)$$

These maps are described by $U(a, \alpha)$ (3.24) with u_i^j identified either with τ_i^j or with ρ_i^j .

Analogously, one proceeds for conjugation σ and involution \dagger which are antilinear (*i.e.*, conjugating complex numbers) counterparts of automorphism and antiautomorphism, respectively,

$$\mathcal{S}(F(\alpha)) = \bar{F}(\sigma(\alpha)), \quad (F(\alpha))^\dagger = \bar{F}(\alpha^\dagger), \quad (3.37)$$

where \bar{F} is complex conjugated to F , *i.e.*, the coefficients of the expansion in powers of $\sigma(\alpha)$ and α^\dagger are complex conjugated to those of the expansion in powers of α .

Consider maps (3.25) with

$$f^i(a) t_i = f(a_n), \quad a_n = \underbrace{a \star \dots \star a}_n, \quad a_1 = a = t_i a^i, \quad a_0 = e_\star, \quad (3.38)$$

where $f(a_n)$ is a linear function of a_n ($n \geq 1$). Such maps have the form (3.25) since $a_n \in A$.

A particularly important subclass of maps (3.25) is represented by \mathbf{U}_u of the form

$$\mathbf{U}_u(a) = \exp[u(a) - a], \quad (3.39)$$

$a \in A$ and

$$u(a) = (u_1^1 a + u_1^2 e_\star) \star (u_2^1 a + u_2^2 e_\star)_\star^{-1}, \quad (e_\star + \beta a)_\star^{-1} := \sum_{n=0}^{\infty} (-\beta)^n a_n \quad (3.40)$$

with $u_i^j \in \mathbb{K}$. Composition of such maps gives a map of the same class

$$\mathbf{U}_u \mathbf{U}_v = \mathbf{U}_{uv}, \quad (3.41)$$

where $(uv)_i^j = u_i^k v_k^j$ is the matrix product in $Mat_2(\mathbb{K})$. The maps \mathbf{U}_u with $\det|u| \neq 0$ are invertible and form usual Mobius group.

From Eq. (3.27) it follows that

$$\mathbf{U}_u(G_\nu) = G_{u(\nu)}. \quad (3.42)$$

Consider the composition law of $M(A)$ in the basis associated with $G_{u(\nu)}$, assuming that new basis elements, that replace (3.4), are

$$T_{i_1 \dots i_n}^u = \frac{\partial^n}{\partial \nu^{i_1} \dots \partial \nu^{i_n}} G_{u(\nu)} \Big|_{\nu=0}. \quad (3.43)$$

To this end, we have to compute

$$G_\nu \diamond G_\mu = \mathbf{U}_u^{-1}(G_{u(\nu)} \circ G_{u(\mu)}). \quad (3.44)$$

Eq. (3.16) gives

$$G_\nu \diamond G_\mu = G_{u^{-1}(u(\nu) \bullet u(\mu))}. \quad (3.45)$$

Generally, maps (3.39) are not filtered, not respecting the condition (3.26). The subgroup P of filtered maps (3.39) is represented by lower triangular matrices

$$u_{b,\beta}(f) = bf \star (e_\star + \beta f)_\star^{-1} \quad (3.46)$$

with the composition law

$$b_{1,2} = b_1 b_2, \quad \beta_{1,2} = \beta_2 + \beta_1 b_2. \quad (3.47)$$

Clearly, P is isomorphic to the affine group of translations and dilatations of \mathbb{R}^1 .

For affine transformations (3.46) we will use notation $U_{b,\beta}$ instead of U_u . In these terms, the unit element is

$$\mathbf{Id} = \mathbf{U}_{1,0} \quad (3.48)$$

and

$$\mathbf{U}_{b,\beta}^{-1} = \mathbf{U}_{b^{-1}, -\beta b^{-1}}. \quad (3.49)$$

The map

$$\mathbf{R} = \mathbf{U}_{-1,1} \quad (3.50)$$

is involutive

$$\mathbf{R}^2 = \mathbf{Id} \quad (3.51)$$

and describes an antiautomorphism of $M(A)$. Indeed, one can check (3.32) using that

$$\mathbf{R}(G_\nu) \circ \mathbf{R}(G_\mu) = \exp[(e_\star + \nu)_\star^{-1} - e_\star] \bullet ((e_\star + \mu)_\star^{-1} - e_\star) \quad (3.52)$$

and, by (3.19),

$$\mathbf{R}(G_\mu \circ G_\nu) = \exp((e_\star + \mu \bullet \nu)_\star^{-1} - e_\star). \quad (3.53)$$

Eq. (3.42) gives

$$\mathbf{R}(G_\nu) = G_{-\nu_\star(e_\star + \nu)_\star^{-1}}. \quad (3.54)$$

Differentiation over ν^i gives, in particular,

$$\mathbf{R}(Id) = Id, \quad (3.55)$$

$$\mathbf{R}(\alpha_i) = -\alpha_i, \quad (3.56)$$

$$\mathbf{R}(\alpha_i \alpha_j) = \alpha_i \alpha_j + \{\alpha_i, \alpha_j\}_\star, \quad (3.57)$$

where, for simplicity, we consider the even case with $\pi(\alpha_i) = 0$.

The antiautomorphism \mathbf{R} of $M(A)$ exists independently of the specific structure of A and is called principal antiautomorphism of $U(l(A))$ [94]. Note that the form of \mathbf{R} (3.39), (3.50) is specific for universal enveloping algebra of a Lie algebra associated with an algebra A .

Given algebra A , the *opposite* algebra \tilde{A} is isomorphic to A as a linear space and has the product law $\tilde{\circ}$

$$a \tilde{\circ} b = b \circ a. \quad (3.58)$$

An antiautomorphism ρ of A can be interpreted as the homomorphism between A and \tilde{A} . If ρ is invertible, \tilde{A} is isomorphic to A . Hence, Eq. (3.51) proves

$$M(\tilde{A}) = \widetilde{M(A)} \sim M(A), \quad \forall A, \quad (3.59)$$

which is of course in agreement with the realization of $M(A)$ as $U(l(A))$ [94].

For affine maps, the composition law (3.45) takes the form

$$G_\nu \diamond G_\mu = G_{\sigma_{b,\beta}(\nu,\mu)}, \quad (3.60)$$

where

$$\sigma_{b,\beta}(\nu,\mu) = -\beta^{-1}(e_\star - (e_\star + \beta\mu) \star (e_\star - \beta(b + \beta)\nu \star \mu)^{-1} \star (e_\star + \beta\nu)). \quad (3.61)$$

In particular, this formula gives

$$\sigma_{1,0}(\nu,\mu) = \nu + \mu + \nu \star \mu = \nu \bullet \mu, \quad (3.62)$$

$$\sigma_{-1,1}(\nu,\mu) = \nu + \mu + \mu \star \nu = \mu \bullet \nu, \quad (3.63)$$

$$\sigma_{1,-\frac{1}{2}}(\nu,\mu) = 2(e_\star - (2e_\star - \mu) \star (4e_\star + \nu \star \mu)_\star^{-1} \star (2e_\star - \nu)). \quad (3.64)$$

Here $\sigma_{1,0}(\nu,\mu)$ corresponds to the unit map, $\sigma_{-1,1}(\nu,\mu)$ corresponds to the antiautomorphism \mathbf{R} , while $\sigma_{1,-\frac{1}{2}}(\nu,\mu)$ describes the map reproducing the $\mathcal{N} \rightarrow 0$ limit of the F -current operator algebra of [35].

3.3 Supertrace and central charge

Let A possess a (super)trace tr obeying

$$tr(a \star b) = (-1)^{\pi(a)\pi(b)} tr(b \star a), \quad \forall a, b \in A \quad (3.65)$$

($\pi(a) = 0$ or 1 is the \mathbb{Z}_2 grading of a ; usual trace is a particular case with $\pi(a) \equiv 0$.) Let A admit such a basis t_i that

$$tr\left(\sum_i a^i t_i\right) = a^0, \quad (3.66)$$

i.e., $tr(t_i) = \delta_i^0$. Then

$$g_{ij} = f_{ij}^0 \quad (3.67)$$

is (graded)symmetric

$$g_{ij} = (-1)^{\pi_i \pi_j} g_{ji}. \quad (3.68)$$

Note that tr is supposed to be even, *i.e.*, t_0 is even which implies that g_{ij} is nonzero if $\pi_i = \pi_j$. If the bilinear form $tr(a \star b)$ is non-degenerate, which is necessarily true if A is simple since zeros of $tr(a \star b)$ form a two-sided ideal of A , g_{ij} can be interpreted as a non-degenerate metric. Associativity of A implies via $tr((a \star b) \star c) = tr(a \star (b \star c))$ graded cyclicity of the structure coefficients

$$f_{ijk} = (-1)^{\pi_i} f_{jki}, \quad f_{ijk} = f_{ijn}^n g_{nk}. \quad (3.69)$$

For unital algebra A , it is convenient to set $t_0 = e_\star$ that is reachable via rescaling of tr in the non-degenerate case with $tr(e_\star) \neq 0$. In the degenerate case with $tr(e_\star) = 0$ a basis element supporting trace differs from e_\star analogously to the case of $psu(2, 2|4)$ familiar from $N = 4$ SYM. In that case, $l(A)$ acquires an ideal associated with tr in addition to that associated with e_\star . Note that, being related to $N = 4$ SYM, the degenerate case may be of primary importance in the multiparticle extension of HS theory.

Trace of A induces a family of traces of $M(A)$. Indeed, for exponentials G_ν (3.18) define trace as

$$Tr_{\Phi, \phi_\star}(G_\nu) = \Phi(tr(\phi_\star(\nu))), \quad \phi_\star(\nu) = \sum_{n=0}^{\infty} \phi_n \underbrace{\nu \star \dots \star \nu}_n, \quad (3.70)$$

with any star-product function $\phi_\star(\nu)$ and usual function $\Phi(x)$. From (3.19) it follows that

$$Tr_{\Phi, \phi_\star}(G_\nu \circ G_\mu) = \Phi(tr(\phi_\star(\nu \bullet \mu))). \quad (3.71)$$

Using (3.65) and (3.16) it is easy to see that $tr(\phi_\star(\nu \bullet \mu)) = tr(\phi_\star(\mu \bullet \nu))$ and, hence,

$$Tr_{\Phi, \phi_\star}(G_\nu \circ G_\mu) = Tr_{\Phi, \phi_\star}(G_\mu \circ G_\nu). \quad (3.72)$$

Since G_ν is the generating function for any element of $M(A)$, formula (3.70) defines a trace of $M(A)$. Thus the space of traces of $M(A)$ admits at least an ambiguity in two functions of one variable. Note that the ambiguity in the definition of trace in $M(A)$ reflects the fact that $M(A)$ is not simple as is the case for every universal enveloping algebra.

Remarkably, the ambiguity of the definition of Tr of $M(A)$ is closely related to the ambiguity in the central charge of the current operator algebra. To reproduce the dependence on the central charge the basis has to be modified further by virtue of a field redefinition of the form

$$G_\nu \rightarrow \eta(\nu)G_\nu, \quad (3.73)$$

where $\eta(\nu)$ is some map from A to \mathbb{K} . This map modifies the product law (3.60) to

$$\tilde{G}_\nu \diamond \tilde{G}_\mu = \frac{\tilde{\eta}(\mu)\tilde{\eta}(\nu)}{\tilde{\eta}(\sigma_{b,\beta}(\nu, \mu))} \tilde{G}_{\sigma_{b,\beta}(\nu, \mu)}, \quad \tilde{\eta}(\nu) = \eta(u_{b,\beta}(\nu)). \quad (3.74)$$

The form of the current OPE gets modified since the basis is still defined by the formula analogous to (3.43) with respect to \tilde{G}_ν .

As shown in Section 4, in the case of F -current algebra, the map (3.73) is defined so that $Tr_{\Phi, \phi_\star}(\tilde{G}_\nu) = 1$ for certain $\Phi(x)$ and ϕ_\star . In the case of A -current algebra, the appropriate field redefinition is still of the form (3.73), but it is not directly related with the rescaling of some trace of $M(A)$.

3.4 Ideals and quotients

3.4.1 Ideals induced by (anti)automorphism

Let τ be an automorphism of A . A set of elements that obey

$$\tau(a) = a \quad (3.75)$$

forms a subalgebra A_τ of A . Suppose that τ is involutive, *i.e.*, $\tau^2 = Id$. Then A_τ consists of τ -even elements

$$a = \frac{1}{2}(a + \tau(a)). \quad (3.76)$$

Let some $a \in A_\tau$ have the form

$$a = (b - \tau(b)) \star c, \quad a \in A_\tau, \quad c \in A. \quad (3.77)$$

That $a \in A_\tau$ implies

$$a = \frac{1}{2}(b - \tau(b)) \star (c - \tau(c)). \quad (3.78)$$

Elements (3.77) form a two-sided ideal I_τ of A_τ . Indeed,

$$a \star (b - \tau(b)) = a \star b - \tau(a \star b) \quad \forall a \in A_\tau. \quad (3.79)$$

Eq. (3.78) implies that elements $a = b \star (c - \tau(c))$ form the same ideal I_τ of A_τ . The algebra

$$A^\tau = A_\tau / I_\tau \quad (3.80)$$

is spanned by those elements of A_τ , that cannot be represented as a product of τ -odd elements of A .

In many cases, including usual HS algebras with nontrivial τ , the latter condition turns out to be too strong implying $A^\tau = 0$. For example, this is true for A generated by oscillators Y^A treated as odd elements of the automorphism τ (which is the boson-fermion automorphism for spinorial Y^A). However, in the case of $M(A)$, this construction leads to nontrivial result.

Let ρ be an involutive antiautomorphism of A . As explained in Section 2, depending on a particular choice of ρ , the condition (2.5) singles out the subalgebras $h_o(V)$ or $h_{usp}(V)$ of the HS Lie algebra $h_u(V)$. In the general case let us call them $l_\rho(V)$. Let \mathcal{R} be the antiautomorphism of $M(A)$ associated with ρ via (3.36). Then

$$\mathbf{T} = \mathbf{R}\mathcal{R}, \quad (3.81)$$

where \mathbf{R} is the principal antiautomorphism (3.50), is an automorphism of $M(A)$. It is not hard to see that $M^{\mathbf{T}}(A) \sim U(l_\rho(V))$. Indeed, using (3.56), the condition $\mathbf{T}(\alpha_i) = \alpha_i$ implies in particular

$$\rho(\alpha_i) = -\alpha_i, \quad (3.82)$$

which is just Eq. (2.5). As a result, all ρ -even elements of A do not belong to $M^{\mathbf{T}}(A)$. Factorization of the ideal $\mathbf{I}_{\mathbf{T}}(A)$ takes away the dependence on all ρ -even elements of A . As the algebra of functions of ρ -odd elements of A , $M^{\mathbf{T}}(A) \sim U(l_\rho(A))$.

Using approach of Section 3.2, it is not difficult to obtain explicit formulae for the composition law of $M^{\mathbf{T}}(A)$ in the form analogous to (3.60). Indeed, consider basis (3.42) associated with $G_{u_{1,-1/2}(\nu)}$. Impose the condition

$$\rho(\nu) = -\nu, \quad (3.83)$$

which is nothing else as the factorization condition removing dependence on ρ -even elements. It is not difficult (but fun) to see that so defined elements $G_{u_{1,-1/2}(\nu)}$ are \mathbf{T} invariant, *i.e.*,

$$\mathbf{T}G_{u_{1,-1/2}(\nu)} = G_{u_{1,-1/2}(\nu)} \quad (3.84)$$

which property relies on the identity $\frac{2z}{(1-z)} = -1 + \frac{1+z}{2-(1+z)}$. This allows us to use $\tilde{G}_\nu = G_{u_{1,-1/2}(\nu)}$ as the generating function for elements of $M^{\mathbf{T}}(A)$. Composition law (3.60) gives

$$\tilde{G}_\nu \diamond \tilde{G}_\mu = \tilde{G}_{\sigma_{1,-\frac{1}{2}}(\nu,\mu)}. \quad (3.85)$$

Remarkably, $\sigma_{1,-\frac{1}{2}}(\nu,\mu)$ obeys (3.83) provided that ν and μ do as one can easily see expanding $\sigma_{1,-\frac{1}{2}}(\nu,\mu)$ (3.64) in power series. Hence, the composition law (3.60), (3.64) gives directly the composition law in $M^{\mathbf{T}}(A)$. (A priori, it could happen that the composition of two generating functions gives a generating function that does not respect (3.83), hence requiring factorization of elements of the ideal that might complicate the problem enormously.) Note that, in particular, these formulae provide a simple realization of the universal enveloping algebras of orthogonal and symplectic Lie algebras since the latter are subalgebras of gl_n extracted by the antiautomorphisms ρ generated by symmetric and antisymmetric bilinear forms, respectively.

In fact, the current operator algebra of [35] is associated with $M^{\mathbf{T}}(A)$ where the antiautomorphism ρ of A is defined as $\rho f(Y^A) = i^{\pi(f)} f(iY^A)$. Indeed, it is well-known that nontrivial conserved currents J_s^{ij} of (odd)even spins are (anti)symmetric in their color indices i, j . As explained in [35], this happens just because they obey the condition $\rho(J_s^{ij}) = -J_s^{ij}$.

3.4.2 Ideals induced by central elements

Let \mathcal{C}_a be a basis of the centrum $C(A)$ of A , that forms a subset of t_i . In terms of structure coefficients (3.1), this implies

$$f_{aj}^i = f_{ja}^i. \quad (3.86)$$

By virtue of (3.6), elements $h(\mathcal{C}, Id) \in M(A)$ are central in $M(A)$. Any $h(\mathcal{C}, Id)$ generates a two-sided ideal of $M(A)$. In particular, ideals I_{c_a}

$$F(t) \in I_{c_a} : \quad F(t) = \prod_a (\mathcal{C}_a - c_a Id) G(t), \quad G(t) \subset M(A), \quad (3.87)$$

as well as the quotient algebras $M_{c_a}(A) = M(A)/I_{c_a}$, are parametrized by $c_a \in \mathbb{K}$.

A particularly important case is where A is a unital algebra and $\mathcal{C} = e_\star \in A$. Then $M_c(A)$ is parametrized by a single parameter c resulting from factorization of elements proportional to $e_\star - cId$

$$M_c : \quad e_\star - cId \sim 0. \quad (3.88)$$

In the case where trace is supported by e_\star , the shift $e_\star \rightarrow e_\star + cId$ gives the composition law

$$F(\alpha) \circ G(\alpha) = F(\alpha) \exp \left(\frac{\overleftarrow{\partial}}{\partial \alpha_i} (f_{ij}^n \alpha_n + c g_{ij}) \frac{\overrightarrow{\partial}}{\partial \alpha_j} \right) G(\alpha). \quad (3.89)$$

In the shifted variables, $M_c(A)$ results from dropping e_\star in all formulae. As a result, Eq. (3.89) where F and G depend only on traceless α , describes the composition law in $M_c(A)$.

In the context of HS theories, relevance of algebras $M_c(A)$ is not clear, however, since, as explained in Section 3.5, the factorization (3.88) identifies the physical vacuum (no particles) with the lowest energy state of the space of single-particle states.

3.4.3 Finite-order quotients

Within infinite zoo of ideals of $M(A)$ we will be particularly interested in those that lead to quotient algebras realized by a finite number of tensor products of A . Given function $\Phi(f_n, tr(f_m))$, the span of elements of the form

$$\sum_\alpha \Phi(f_n^\alpha, tr(f_m^\alpha)) \circ G^\alpha, \quad \forall f^\alpha \in A, G^\alpha \in M(A) \quad (3.90)$$

forms a two-sided ideal \mathcal{I}_Φ of $M(A)$. Indeed, since various $h \in A$ generate $M(A)$ it suffices to show that

$$h \circ \mathcal{I}_\Phi \in \mathcal{I}_\Phi, \quad h \in A. \quad (3.91)$$

By virtue of (3.6) we observe

$$h \circ \Phi(f_n, tr(f_m)) - (-1)^{\pi(h)\pi(\Phi)} \Phi(f_n, tr(f_m)) \circ h = \sum_k [h, f_k]_\star \frac{\partial}{\partial f_k} \Phi(f_n, tr(f_m)). \quad (3.92)$$

Using that $[h, \dots]_\star$ is a derivation and that $\text{tr}([h, \dots]_\star) = 0$, we obtain

$$h \circ \Phi(f_n, \text{tr}(f_m)) = (-1)^{\pi(h)\pi(\Phi)} \Phi(f_n, \text{tr}(f_m)) \circ h + \frac{\partial}{\partial \lambda} \Phi(f_n(\lambda), \text{tr}(f_m(\lambda))) \Big|_{\lambda=0}, \quad (3.93)$$

where

$$f(\lambda) = f + \lambda[h, f]_\star, \quad f_n(\lambda) = \underbrace{f(\lambda) \star \dots \star f(\lambda)}_n. \quad (3.94)$$

This proves (3.91) since the *r.h.s.* of Eq. (3.93) can be represented as a linear combination of polynomials $\Phi(f_n, \text{tr}(f_m))$ with different f .

Naively, factorization over order $n + 1$ polynomials $\Phi(f) = (f)^{n+1} + \dots$ should give an algebra spanned by order n polynomials. However, in most cases, this is not true because the ideal \mathcal{I}_Φ turns out to be much larger, coinciding with $M'(A)$. Indeed, consider for example bilinear $\Phi_\gamma(f_n)$

$$\Phi_\gamma(f_n) = f^2 + 2\gamma f \star f \quad (3.95)$$

with an arbitrary parameter γ . In this case, elements

$$(fg + \gamma\{f, g\}_\star) \circ h = fgh + f(g \star h) + g(f \star h) + \gamma\{f, g\}_\star h + \gamma\{f, g\}_\star \star h \quad (3.96)$$

belong to $\mathcal{I}_{\Phi_\gamma}$. Obviously,

$$f(g \star h) + g(f \star h) \sim -\gamma(f \star g \star h + g \star h \star f + g \star f \star h + f \star h \star g), \quad (3.97)$$

$$\{f, g\}_\star h \sim -\gamma(\{f, g\}_\star \star h + h \star \{f, g\}_\star), \quad (3.98)$$

were equivalence is up to terms that belong to $\mathcal{I}_{\Phi_\gamma}$. This gives

$$\begin{aligned} (fg + \gamma\{f, g\}_\star) \circ h &\sim fgh - \gamma(f \star g \star h + g \star f \star h + g \star h \star f + f \star h \star g) \\ &\quad - \gamma^2(\{f, g\}_\star \star h + h \star \{f, g\}_\star) + \gamma\{f, g\}_\star \star h \\ &= fgh - \gamma(g \star h \star f + f \star h \star g) - \gamma^2(\{f, g\}_\star \star h + h \star \{f, g\}_\star). \end{aligned} \quad (3.99)$$

Antisymmetrization of this expression with respect to h and g gives

$$\gamma(1 - \gamma)[f, [h, g]_\star]_\star \in \mathcal{I}_{\Phi_\gamma}. \quad (3.100)$$

Hence, except for $\gamma = 0$ or $\gamma = 1$, $\mathcal{I}_{\Phi_\gamma}$ contains all elements of A that can be represented as $[f, [h, g]_\star]_\star$, *i.e.*, belong to the ideal $l_{(2)}(A)$ of $l(A)$. In the relevant cases where $l_{(1)}(A)$ ($h \in l_{(1)}(A) : h = [f, g]_\star$ for some $f, g \in l(A)$), and hence $l_{(2)}(A)$, is simple, it coincides with almost all $l(A)$. Namely, in the cases of interest $l(A) = l_c(A) \oplus l_{(1)}(A)$ where $l_c(A)$ is the Abelian algebra spanned by central elements of A . (This is fully analogous to the relation $l(\text{Mat}_n(\mathbb{C})) = gl_n(\mathbb{C}) = sl_n(\mathbb{C}) \oplus \mathbb{C}$, where $l_{(1)}(\text{Mat}_n(\mathbb{C})) = sl_n(\mathbb{C})$.) Since $\mathcal{I}_{\Phi_\gamma}$ is too large for generic γ , we consider the special cases of $\gamma = 0$ or $\gamma = 1$.

Obviously, $M(A)/\mathcal{I}_{\Phi_0} = A \oplus \mathbb{K}$. On the other hand, from (3.57) it follows that the case of $\gamma = 1$ is related to $\gamma = 0$ by the principal antiautomorphism \mathbf{R} , *i.e.*, $M(A)/\mathcal{I}_{\Phi_1} = \tilde{A} \oplus \mathbb{K}$. Hence the cases of $\gamma = 0$ and $\gamma = 1$ are exchanged via exchange of A with \tilde{A}

$$M(A)/\mathcal{I}_{\Phi_\gamma} \sim M(\tilde{A})/\mathcal{I}_{\Phi_{1-\gamma}}, \quad \gamma = 0, 1. \quad (3.101)$$

As explained in Section 3.4.2, the unit elements $e_\star \in A$ and $Id \in M(A)$ can be identified via factorization of the ideal I_c . However, in the quotient algebras $M(A)/\mathcal{I}_{\Phi_\gamma}$ the central charge c is no longer arbitrary. Indeed,

$$e_\star \circ f = e_\star f + e_\star \star f \sim f - 2\gamma f = (1 - 2\gamma)f = (1 - 2\gamma)Id \circ f. \quad (3.102)$$

Hence, the factorization over both I_c and $\mathcal{I}_{\Phi_\gamma}$ is possible at $c(\gamma) = (1 - 2\gamma)$, *i.e.*, $c(0) = 1$ and $c(1) = -1$.

The example of \mathcal{I}_{Φ_0} admits natural generalization to the ideals \mathcal{I}^{N+1} generated by Eq. (3.90) with

$$\Phi = f^{N+1}. \quad (3.103)$$

As a linear space, the quotient algebra $M_N(A) := M(A)/\mathcal{I}^{N+1}$ is

$$M_N(A) = \sum_{n=0}^N \oplus Sym \underbrace{A \otimes \dots \otimes A}_n. \quad (3.104)$$

Here the only possible value of c is

$$c_N = N. \quad (3.105)$$

Indeed,

$$e_\star \circ f^N = e_\star f^N + N(e_\star \star f)f^{N-1} \sim Nf^N \quad (3.106)$$

since $e_\star f^N \in \mathcal{I}^{N+1}$. Now one can consider quotient algebras $M^N(A) = M_N(A)/\mathcal{I}_{e_\star - N Id}$. Note that $M^1(A) = A$. Similarly, the generalization of \mathcal{I}_{Φ_1} to higher N leads to ideals \mathcal{I}^{N+1} of $M(\tilde{A})$ and quotients $M_N(\tilde{A})$ and $M^N(\tilde{A})$.

3.5 Modules

Since $M(A) \sim U(l(A))$, any A -module generates a $M(A)$ -module. Tensor product of any number of A -modules forms a $l(A)$ -module and, hence, $M(A)$ -module. Beyond that, $M(A)$ admits less trivial modules which may be relevant in the context of multiparticle HS theories.

Let V be an A -module. Recall that in the HS context $A = H_{V_\Phi}$ and $V = V_\Phi$ is the space of single-particle states of some fields Φ . The space of all multiparticle states is

$$\mathcal{V} = \sum_{n=0}^{\infty} \oplus V^n, \quad V^n = Sym \underbrace{V \otimes \dots \otimes V}_n. \quad (3.107)$$

When referring to a particular field theory associated with V_Φ we will use notation \mathcal{V}_Φ .

Let τ_α be a basis of V

$$v \in V : \quad v = \sum_{\alpha} v^\alpha \tau_\alpha \quad (3.108)$$

and

$$t_i(\tau_\alpha) = T_{i\alpha}^\beta \tau_\beta, \quad (t_i \star t_j)(\tau_\alpha) = T_{j\alpha}^\gamma T_{i\gamma}^\beta \tau_\beta \longrightarrow T_{j\alpha}^\gamma T_{i\gamma}^\beta = f_{ij}^k T_{k\alpha}^\beta. \quad (3.109)$$

Let V^* be dual to V

$$\lambda \in V^* : \quad \lambda = \sum_{\beta} \lambda_{\beta} \tau^{*\beta}. \quad (3.110)$$

Similarly to the realization of $M(A)$ in terms of functions $F(\alpha)$, elements of \mathcal{V} can be represented by functions $\phi(\lambda)$ on V^*

$$\phi(\lambda) = \sum_{n=0}^{\infty} \phi^{\alpha_1 \dots \alpha_n} \lambda_{\alpha_1} \dots \lambda_{\alpha_n}. \quad (3.111)$$

Let $F(\alpha) \in M(A)$, $\phi(\lambda) \in \mathcal{V}$. \mathcal{V} can be endowed with the structure of $M(A)$ -module by setting

$$F(\alpha)(\phi(\lambda)) = F(\alpha) \exp \left(\frac{\overleftarrow{\partial}}{\partial \alpha_i} t_i(\lambda_{\beta}) \frac{\overrightarrow{\partial}}{\partial \lambda_{\beta}} \right) \phi(\lambda) \Big|_{\alpha_i = \mathbf{t}_i(\lambda)}, \quad (3.112)$$

where $\mathbf{t}_i(\lambda)$ is some linear function of λ_{α} , that obeys the condition

$$t_i(\lambda_{\beta}) \frac{\partial \mathbf{t}_k(\lambda)}{\partial \lambda_{\beta}} = f_{ik}^j \mathbf{t}_j(\lambda). \quad (3.113)$$

That Eq. (3.112) defines a $M(A)$ -module, *i.e.*,

$$F(t)(G(t)(\phi(\lambda))) = (F \circ G)(\phi(\lambda)), \quad (3.114)$$

is easy to see using Eqs. (3.109), (3.113).

Let $\mathcal{V}(V, \mathbf{t})$ be the $M(A)$ -module \mathcal{V} (3.112) determined by an A -module V and $\mathbf{t}_i(\lambda)$ solving (3.113). The $M(A)$ -module $\mathcal{V}(V, 0)$ is associated with $\mathbf{t}_i = 0$. A less trivial option of

$$\mathbf{t}_i(\lambda) = o(t_i(\lambda)) = o^{\alpha} T_{i\alpha}^{\beta} \lambda_{\beta} \quad (3.115)$$

is parametrized by a vector $o^{\beta} \in V$. Indeed, in this case (3.113) holds by virtue of (3.109).

The module $\mathcal{V}(V, 0)$ is infinitely reducible. Indeed, from Eq. (3.112) with $\mathbf{t}_i = 0$ it follows that the subspace V^p of homogeneous polynomials $\phi(\lambda)$ of degree p remains invariant under the action of $M(A)$. (Note that existence of the ideals \mathcal{I}^{N+1} (3.103) is closely related to the fact of reducibility of $\mathcal{V}(V, 0)$: \mathcal{I}^{N+1} is the annihilator of V^p with $p \leq N$.) Clearly, V^p are canonical $U(l(A))$ -modules associated with symmetrized tensor products of the $l(A)$ -module V , *i.e.*, spaces of p -particle states in their multiparticle interpretation.

For $\mathbf{t}_i \neq 0$ (3.115), the action of $M(A)$ (3.112) is not homogeneous, mixing V^n with different n . Suppose that V is induced from the vacuum vector o , *i.e.*, $A(o) = V$. Let V^* be the right A -module and $o^* \in V^*$ be normalized so that $o^*(o) = 1$. From Eqs. (3.112), (3.115) it follows that, in this case, the $M(A)$ -module $\mathcal{V}(V, \mathbf{t})$ is induced from the vacuum element $O \in \mathcal{V}$ identified with $\phi(\lambda) = 1$. Modules of this type are somewhat analogous to Fock modules of oscillator algebra (as is illustrated in the next section by the example of Weyl algebra) and are expected to play an important role in multiparticle theories.

$M_c(A)$ (3.88) results from $M(A)$ via factorization of elements proportional to $e_{\star} - cId$. Similarly, a $M_c(A)$ -module $\mathcal{V}_c(V, \mathbf{t})$ results from $\mathcal{V}(V, \mathbf{t})$ via factorization of elements induced from

$$o - cO \sim 0. \quad (3.116)$$

Indeed, according to (3.112), the action of $e_\star - cId$ on O gives

$$(e_\star - cId)(O) = o(\lambda) - cO. \quad (3.117)$$

To reduce the $M(A)$ -module $\mathcal{V}(V, \mathbf{t})$ to the $M_c(A)$ -module $\mathcal{V}_c(V, \mathbf{t})$ using relation (3.116), one has to remove the dependence on λ_o from $\lambda = \lambda_o o^\star + \sum_\beta \tilde{\lambda}_\beta \tilde{\tau}^\beta$. In other words, $\mathcal{V}_c(V, \mathbf{t})$ consists of functions of all elements of \mathcal{V}^\star except for the vacuum o^\star . Since such a factorization identifies the vacuum o of the space of single-particle space, usually describing the lowest energy state of one or another particle, with the physical vacuum O with no particles, its physical meaning is however obscure. In fact, the difficulties of the naive extension of HS algebras discussed in Introduction resulted just from consideration of $M_c(A)$ -modules instead of $M(A)$ -modules.

Another construction applicable to a Lie algebra $l(A)$ with any A , which is particularly useful in the context of HS theory, is that of twisted adjoint modules. Let τ be some automorphism of A . The τ -twisted adjoint l_A -module A_τ has A as a linear space where l_A acts as follows

$$a(b) = a \star b - b \star \tau(a), \quad a \in l_A, \quad b \in A_\tau. \quad (3.118)$$

Any τ -twisted adjoint module A_τ of $l(A)$ admits straightforward extension to \mathcal{T} -twisted adjoint module of $l(M(A))$. This simple observation is expected to play a key rôle for the formulation of a multiparticle generalization of HS gauge theory.

3.6 Weyl algebra and Fock module

Weyl algebra A_M , which underlies the construction of most of HS algebras, is the unital algebra generated by $2M$ elements Y_A satisfying (1.1). Remarkably, it can itself be interpreted as the quotient of a multiparticle algebra $M(a_M)$. Here a_M is the algebra with the generating elements Y_Ω and h obeying relations

$$Y_A \star Y_B = K_{AB}h, \quad Y_A \star h = h \star Y_A = 0, \quad h \star h = 0, \quad (3.119)$$

where $A, B = 1, \dots, M$, and K_{AB} is some matrix with nondegenerate antisymmetric part

$$C_{AB} = (K_{AB} - K_{BA}). \quad (3.120)$$

Algebra a_M is obviously associative since any triple product of its elements vanishes.

$M(a_M)$ is spanned by functions $f(Y_A, h)$. This is not yet Weyl algebra, but rather the algebra of quantum operators in the deformation quantization framework with h interpreted as a deformation parameter. Weyl algebra $A_M = M_{\hbar}(a_M)$ results from $M(a_M)$ via factorization of the ideal generated by $h - \hbar Id$ with parameter \hbar . Note that $M_{\hbar}(a_M)$ with various $\hbar \neq 0$ are pairwise isomorphic. The “classical” case of $\hbar = 0$ is degenerate.

Different choices of the symmetric part of $K_{\Omega\Lambda}$ lead to different product laws (3.6) which correspond to different star products for the same Weyl algebra. Indeed, it is well known that different choices of $K_{\Omega\Lambda}$ with the same $C_{\Omega\Lambda}$ (3.120) encode different ordering prescriptions.

Let us now explain how Fock module of Weyl algebra results from the construction of Section 3.5. Consider for simplicity the case of a_1 with the defining relations

$$Y_- \star Y_+ = h, \quad Y_+ \star Y_- = 0, \quad Y_\pm \star Y_\pm = 0, \quad Y_\pm \star h = h \star Y_\pm = 0, \quad h \star h = 0. \quad (3.121)$$

A left a_1 -module in a two-dimensional vector space V with basis elements v and v_+ , which can be realized as a quotient of the left adjoint a_1 -module, is

$$Y_-v = 0, \quad Y_-v_+ = v, \quad Y_+v = 0, \quad Y_+v_+ = 0, \quad hv = 0, \quad hv_+ = 0. \quad (3.122)$$

It is easy to see that

$$\mathbf{t}_-(v) = 0, \quad \mathbf{t}_+(v) = v_+, \quad \mathbf{t}_h(v) = v \quad (3.123)$$

solves Eq. (3.113). With this substitution, Eq. (3.112) gives the $M(a_1)$ -module realized by functions $\Phi(v_+, v)$ with

$$\alpha_-(\Phi) = v \frac{\partial}{\partial v_+} \Phi, \quad \alpha_+(\Phi) = v_+ \Phi, \quad \alpha_h(\Phi) = v \Phi. \quad (3.124)$$

Factorization of the ideal of $M(a_1)$ generated by $h - \hbar Id$ along with its image in the constructed $M(a_1)$ -module implies the substitution $v \rightarrow \hbar$, giving the Fock module of the Weyl algebra A_1 .

4 Current operator algebra

In this section we show how the current operator algebra in the twistor space results from our construction. The space-time current operator algebra, which results from the twistor one via unfolded formulation of the current conservation equations, is not considered in this paper. We refer the reader to [35] on the details of this relation.

The dictionary between notations of this paper and [35] is as follows. Free currents $\mathcal{J}^2(U, V)$, where U and V denote twistor variables used in [35], identify with the generators t_i or, equivalently, with the basis α_i of A^* . The normal ordered product $:\mathcal{J}^2(U_1, V_1) \dots \mathcal{J}^2(U_n, V_n):$, which is by construction symmetric with respect to the permutation of arguments (U_b, V_b) with different b , is represented by the usual product $t_{i_1} \dots t_{i_n}$. Parameters ν^i have to be identified with the parameters of currents called $g(W_1, W_2)$ in [35] and usual powers $(\nu)^n$ represent $\mathcal{J}_g^{2n} =: (\mathcal{J}_g^2)^n$.

The currents considered in [35] are invariant under certain involutive operation μ as a consequence of the construction of currents in terms of bilinears of free fields. In the setup of this paper $\mu = -\rho$, where ρ is the antiautomorphism of Section 3. The current algebra of [35] is nothing else as the quotient algebra $M^{\mathbf{T}}(A)$ introduced in Section 3.4.1 where \mathbf{T} is the antiautomorphism of $M(A)$ generated by ρ . As explained in Section 3.4.1, the specific form of the composition law associated with $\sigma_{1, -\frac{1}{2}}$ is compatible with the (factorization) condition $\rho(\nu) = -\nu$ imposed in [35].

In the case of A -currents with stripped indices referring to different frequencies of constituent fields, the current operator algebra is most conveniently formulated in terms of bi-associative algebra A endowed with two mutually associative products.

4.1 F -current algebra

As mentioned in the end of Section 3.4.1, F -current algebra at $\mathcal{N} = 0$ is described by the composition law (3.85). To describe \mathcal{N} -dependent terms, one should generalize it using (3.73)

with appropriate function $\eta(\nu)$. For

$$\eta(\nu) = Tr_{\Phi, \phi_\star}(G_{u_{1,\beta}(\nu)}), \quad \Phi(x) = \exp -x \quad (4.1)$$

with some $\phi_\star(\nu)$, the factor on the *r.h.s.* of (3.74) is

$$\exp tr \left(\phi_\star(u_{1,\beta}(\nu) \star u_{1,\beta}(\mu) + u_{1,\beta}(\nu) + u_{1,\beta}(\mu)) - \phi_\star(u_{1,\beta}(\mu)) - \phi_\star(u_{1,\beta}(\nu)) \right). \quad (4.2)$$

The characteristic property of the F -current operator algebra is that the trace-dependent part of the OPE for both right and left multiplication with the bilinear current \mathcal{J}_μ^2 only involves the trace $tr(\nu \star \mu)$ between parameters of two elementary currents \mathcal{J}_ν^2 and \mathcal{J}_μ^2 . This imposes the condition that the part of (4.2) linear either in μ or in ν should have the form $\frac{1}{8}\mathcal{N}tr(\nu \star \mu)$. This gives the differential equation

$$\phi'_\star(u_{1,\beta}(\nu)) \star (u_{1,\beta}(\nu) + e_\star) = \frac{1}{8}\mathcal{N}\nu \quad (4.3)$$

solved by

$$\phi_\star(u_{1,\beta}(\nu)) = \frac{1}{8}\mathcal{N} \left(\beta^{-1} \ln_\star(e_\star + \beta\nu) - (1 + \beta)^{-1} \ln_\star(e_\star + (1 + \beta)\nu) \right). \quad (4.4)$$

For $\beta = -\frac{1}{2}$, this gives

$$\phi_\star(u_{1,\beta}(\nu)) = -\frac{1}{4}\mathcal{N} \ln_\star(e_\star - \frac{1}{4}\nu \star \nu). \quad (4.5)$$

Introducing

$$\tilde{G}_\nu = \eta(\nu) G_{u_{1,-\frac{1}{2}}(\nu)}, \quad (4.6)$$

the trace-dependent version of formula (3.60) at $b = 1$, $\beta = -1/2$ takes the form

$$\tilde{G}_\nu \diamond \tilde{G}_\mu = \left(\frac{det_\star |e_\star - \frac{1}{4}\nu \star \nu| det_\star |e_\star - \frac{1}{4}\mu \star \mu|}{det_\star |e_\star - \frac{1}{4}\sigma_{1,-\frac{1}{2}}(\nu, \mu) \star \sigma_{1,-\frac{1}{2}}(\nu, \mu)|} \right)^{\frac{\mathcal{N}}{4}} \tilde{G}_{\sigma_{1,-\frac{1}{2}}(\nu, \mu)}, \quad (4.7)$$

where, as usual,

$$det_\star |A| = \exp tr(\ln_\star(A)). \quad (4.8)$$

(Of course, the \star -determinant possesses the multiplication property $det_\star |A \star B| = det_\star |A| det_\star |B|$.)

Formula (4.7) gives the generating function for the F -current operator algebra of [35]. To see this it remains to check the parts of $\sigma_{1,-\frac{1}{2}}(\nu, \mu)$ linear either in ν or in μ which describe left and right multiplication with \mathcal{J}_ν^2 and \mathcal{J}_μ^2 , respectively. Once, formula (4.7) correctly reproduces this part of the algebra, associativity implies that it describes the full operator algebra.

For example, denoting by \tilde{G}_μ^2 the part of \tilde{G}_μ linear in μ , we obtain from (4.7)

$$\tilde{G}_\nu \diamond \tilde{G}_\mu^2 = \tilde{G}_\nu \left(\mu + \frac{1}{2}(\nu \star \mu - \mu \star \nu) - \frac{1}{4}\nu \star \mu \star \nu + \frac{1}{8}\mathcal{N}tr(\nu \star \mu)Id \right). \quad (4.9)$$

These terms reproduce OPE of $\mathcal{J}_\nu^{2n} \mathcal{J}_\mu^2$. Indeed, the first term is the regular one. The second results from single contractions of the constituent fields. The third term results from double contractions of the constituent fields of \mathcal{J}_μ^2 with two different \mathcal{J}_ν^2 while the last one comes from the double contraction of the constituent fields of \mathcal{J}_μ^2 with those of some \mathcal{J}_ν^2 .

Formula, (4.7) represents OPE of $\mathcal{J}_\nu^{2n} \mathcal{J}_\mu^{2m}$ by the n - and m -linear terms in ν and μ . Note that being represented by tr , the \mathcal{N} -dependent central term does not contribute to the commutator $\mathcal{J}_\nu^2 \mathcal{J}_\mu^{2m} - \mathcal{J}_\mu^{2m} \mathcal{J}_\nu^2$. This is because the dependence on \mathcal{N} was introduced in (4.1) in terms of a trace of $M(A)$.

4.2 A -current algebra

A -current algebra is the algebra of currents with stripped indices $a, b = +, -$ associated with the creation and annihilation parts of the constituent fields. This algebra is anticipated to play a role in the analysis of amplitudes (hence A -current algebra). Its structure differs from that of F -current algebra in several respects. In particular, the part of OPE associated with unity does contribute to the operator commutator. Hence it cannot be derived via a field redefinition (3.73) where η is expressed in terms of some trace of $M(A)$ as in (4.1), requiring η of some other form. Another novelty is that A -operator algebra involves two associative products instead of one in the F -current case. This bi-associative structure underlies the construction of butterfly product of [35]. In this section, we first describe the relevant bi-associative structure and then present explicit formulae for the A -current operator algebra.

4.2.1 Bi-associative algebra

By *bi-associative* algebra we mean a linear space A over a field \mathbb{K} endowed with two mutually associative products \star and \cdot . This implies that A is an associative algebra with respect to the product $\alpha \star + \beta \cdot$ with any $\alpha, \beta \in \mathbb{K}$. Equivalently, in addition to associativity of \star and \cdot ,

$$(a \star b) \cdot c = a \star (b \cdot c), \quad (a \cdot b) \star c = a \cdot (b \star c). \quad (4.10)$$

Biassociative algebras can be easily introduced as follows. Choose two elements τ_1, τ_2 of some associative algebra A with a product law $*$. Then the two products $a \star b := a * \tau_1 * b$ and $a \cdot b := a * \tau_2 * b$ are associative along with any their linear combination, endowing A with the bi-associative structure.

A is assumed to be unital with respect to both \star and \cdot . However, the respective units e_\star and e_\cdot may be different. They satisfy the following obvious relations

$$e_\cdot \cdot e_\cdot = e_\cdot, \quad e_\star \star e_\star = e_\star, \quad e_\cdot \cdot e_\star = e_\star \cdot e_\cdot = e_\star, \quad e_\cdot \star e_\star = e_\star \star e_\cdot = e_\cdot. \quad (4.11)$$

One can see that, in addition, via relative rescaling of the two products (and, hence, respective units) it is possible to achieve that

$$e_\cdot \star e_\cdot = e_\star, \quad e_\star \cdot e_\star = e_\cdot. \quad (4.12)$$

In the sequel, (4.12) is assumed to be true. These relations imply in particular

$$a \cdot b = a \star e \star b, \quad a \star b = a \cdot e_\star \cdot b \quad (4.13)$$

and that

$$\Pi_\pm = \frac{1}{2}(e_\star \pm e.) \quad (4.14)$$

are projectors with respect of each of the products

$$\Pi_\pm \star \Pi_\pm = \Pi_\pm \cdot \Pi_\pm = \Pi_\pm, \quad \Pi_\pm \star \Pi_\mp = \Pi_\pm \cdot \Pi_\mp = 0. \quad (4.15)$$

Inverse elements are defined as usual

$$a_\star^{-1} \star a = a \star a_\star^{-1} = e_\star, \quad a.^{-1} \cdot a = a \cdot a.^{-1} = e. \quad (4.16)$$

The property (4.12) has the following nice consequences

$$a_\star^{-1} = e_\star \cdot a.^{-1} \cdot e_\star, \quad a.^{-1} = e. \star a_\star^{-1} \star e., \quad (4.17)$$

$$(a \cdot b)_\star^{-1} = b_\star^{-1} \cdot a_\star^{-1}, \quad (a \star b).^{-1} = b.^{-1} \star a.^{-1}. \quad (4.18)$$

Note that from here it follows that e_\star and $e.$ coincide with their inverses with respect to each of the products

$$(e_\star).^{-1} = e_\star, \quad (e.)_\star^{-1} = e. \quad (4.19)$$

In A -current operator algebra an important role is played by an involutive atiautomorphism ρ obeying

$$\rho(a \star b) = \rho(b) \star \rho(a), \quad \rho^2 = Id, \quad \rho(e_\star) = e_\star, \quad \rho(e.) = -e., \quad \rho(\Pi_\pm) = \Pi_\mp. \quad (4.20)$$

By virtue of (4.13), this implies

$$\rho(a \cdot b) = -\rho(b) \cdot \rho(a). \quad (4.21)$$

The products \triangleright and \triangleleft of [35] are

$$a \triangleright b = -a \star \Pi_+ \star b, \quad a \triangleleft b = a \star \Pi_- \star b. \quad (4.22)$$

4.2.2 Bi-maps and OPE

Starting from the algebra A endowed with the product law \star , we can now apply the linear map (3.39) with $u(f)$ built with the aid of \cdot . For maps analogous to (3.46)

$$u_{b,\beta}(f) = bf \cdot (e. + \beta f).^{-1} = \frac{b}{\beta} (e. - (e. + \beta f).^{-1}), \quad (4.23)$$

an elementary computation gives again formula (3.60) where

$$\sigma_{b,\beta}(\nu, \mu) = -\beta^{-1} \left(e. - (e. + \beta \mu) \star [e. - (\beta \nu - 2\Pi_-) \star (e. + b\beta^{-1} e_\star) \star (\beta \mu - 2\Pi_-)].^{-1} \star (e. + \beta \nu) \right). \quad (4.24)$$

At $\cdot = \star$, $e_\cdot = e_\star$, this formula reproduces (3.61). The map, that leads to the basis corresponding to the A -current OPE, is a composition of some maps (3.42) and (4.24).

Indeed, from general properties of the current operator algebra it follows that it should be isomorphic to the algebra $M^{\mathbf{T}}(A)$ associated with an appropriate antiautomorphism ρ . Hence, the product law of [35] should result from (3.85) via some field redefinition analogous to (3.39). However, now we should confine ourselves to such field redefinitions that leave $M^{\mathbf{T}}(A)$ invariant. This condition rules out field redefinitions (3.39), (3.40) constructed in terms of the product \star (called \bullet in [35]) because even \star -degrees of elements ν obeying (3.83) do not obey this condition, hence not belonging to $M^{\mathbf{T}}(A)$. The trick is that it is possible to use the product law \cdot , which is ρ -odd itself obeying (4.21), to perform a change of variables (4.23) that maps $M^{\mathbf{T}}(A)$ to itself.

Thus, the map $U_{1,-\frac{1}{2}}^\star$ with respect to the product \star should be followed by some map $U_{b,\beta}$ with respect to \cdot . It turns out that the appropriate form of the current operator algebra results from the map $U_{1,-\frac{1}{2}}^\cdot$. Practically, it is most convenient to apply the map

$$U_{1,-\frac{1}{2}}^\star(G_\nu) = U_{1,-\frac{1}{2}}^\cdot(U_{1,-\frac{1}{2}}^\star(G_\nu)) \quad (4.25)$$

directly to (3.19) rather than to apply $U_{1,-\frac{1}{2}}^\cdot$ to the product law (3.85).

Using formulae of Section 4.2.1, it is not difficult to obtain that

$$U_{1,-\frac{1}{2}}^\star(G_\nu) = G_{\nu\star(e_\star - \Pi_+ \star \nu)_\star}^{-1} \quad (4.26)$$

with Π_+ (4.14). The product law in the new basis is

$$\tilde{G}_\nu \star \tilde{G}_\mu = \tilde{G}_{\sigma_{1,-\frac{1}{2}}^\star(\nu,\mu)}, \quad (4.27)$$

where

$$\sigma_{1,-\frac{1}{2}}^\star(\nu,\mu) = (e_\star + \nu \star \Pi_-) \star (e_\star + \mu \star \Pi_+ \star \nu \star \Pi_-)_\star^{-1} \star \mu + (e_\star - \mu \star \Pi_+) \star (e_\star + \nu \star \Pi_- \star \mu \star \Pi_+)_\star^{-1} \star \nu. \quad (4.28)$$

To reproduce the trace-dependent terms we proceed as in the F -current algebra case, requiring that if such terms are linear in μ , they should also be linear in ν having the form

$$- \mathcal{N} \operatorname{tr}(\Pi_+ \star \nu \star \Pi_- \star \mu). \quad (4.29)$$

This is achieved via (3.74) with

$$\tilde{\eta}(x) = \exp[-\mathcal{N} \operatorname{tr}_\star(\ln_\star(e_\star - \Pi_+ x))], \quad (4.30)$$

giving the composition law

$$\tilde{G}_\nu \diamond \tilde{G}_\mu = \left(\frac{\det_\star |e_\star - \Pi_+ \star \sigma_{1,-\frac{1}{2}}^\star(\nu,\mu)|}{\det_\star |e_\star - \Pi_+ \star \nu| \det_\star |e_\star - \Pi_+ \star \mu|} \right)^\mathcal{N} \tilde{G}_{\sigma_{1,-\frac{1}{2}}^\star(\nu,\mu)}. \quad (4.31)$$

This formula encodes in a concise form the full operator algebra of free $3d$ conserved A -currents. The prefactor on its *r.h.s.* provides the generating function for two-point functions $\langle \mathcal{J}_\nu^{2n} \mathcal{J}_\mu^{2m} \rangle$.

To compare this product with the A -current operator algebra of [35], it remains to consider the part linear in μ

$$\tilde{G}_\nu \diamond \tilde{G}_\mu^2 = \tilde{G}_\nu(\mu + \nu \star \Pi_- \star \mu - \mu \star \Pi_+ \star \nu - \nu \star \Pi_- \star \mu \star \Pi_+ \star \nu - \mathcal{N} \operatorname{tr}(\Pi_+ \star \nu \star \Pi_- \star \mu)). \quad (4.32)$$

Using (4.22), this formula just reproduces OPE for $J_\nu^{2n} J_\mu^2$ of [35].

5 Multiparticle Lie (super)algebras and further extensions

As explained in Section 2, HS algebras in their conformal interpretation describe maximal symmetries of the space of single-particle states of free conformal field theory of some fields Φ . We propose that a multiparticle extension of HS symmetry, that can serve as the symmetry algebra of the space \mathcal{V}_Φ of all multiparticle states of fields Φ , is provided by an appropriate real form of the Lie (super)algebra $l(M(H_{V_\Phi}))$. Specifically, we consider the Lie algebra $m_u(V_\Phi)$ which is the real form of $l(M(H_{V_\Phi}))$ singled out by the condition

$$\mathcal{S}(f) = f, \quad (5.1)$$

where \mathcal{S} is the conjugation of $l(M(H_{V_\Phi}))$ induced via (3.37) by the conjugation σ in the reality conditions (2.2) for $h_u(V_\Phi)$ (see [23, 3, 17]).

Indeed, in accordance with Eq. (3.112), $M(H_{V_\Phi})$ acts on the space \mathcal{V}_Φ (3.107) of all multiparticle states of the field Φ . Hence, the Lie algebra $l(M(H_{V_\Phi}))$ is the complexified symmetry algebra of \mathcal{V}_Φ , while $m_u(V_\Phi)$ represents the appropriate real symmetry of \mathcal{V}_Φ .

Real algebras $m_o(V_\Phi)$ and $m_{usp}(V_\Phi)$ are singled out by the same condition (5.1) from the complex Lie algebras $M^\mathbf{T}(H_{V_\Phi})$ introduced in Section 3.4.1 by virtue of the antiautomorphism ρ used in (2.5) to single out subalgebras $h_o(V_\Phi)$ or $h_{usp}(V_\Phi)$ from $h_u(V_\Phi)$.

As a linear space,

$$H_V = \sum_{m,n=0}^{\infty} (V^m)^* \otimes V^n, \quad (5.2)$$

while

$$M(H_V) \sim \sum_{n=0}^{\infty} (V^n)^* \otimes V^n, \quad m_u(V_\Phi) \sim \sum_{n=0}^{\infty} (V^n)^* \otimes V^n. \quad (5.3)$$

$m_o(V_\Phi)$ and $m_{usp}(V_\Phi)$ are represented by the further symmetrization

$$M(H_V) \sim \sum_{n=0}^{\infty} \text{Sym}((V^n)^* \otimes V^n) \quad (5.4)$$

upon the identification $V^* = \rho(V)$ expressing the condition (2.4). Obviously, $M(H_V) \subset H_V$, *i.e.*, multiparticle extension of a HS algebra is not itself a HS algebra.

It is useful to use the following realization of multiparticle algebras. Consider N copies of generators t_i^α of A ($\alpha = 1 \dots N$), that satisfy

$$t_i^\alpha \star t_j^\alpha = f_{ij}^k t_k^\alpha, \quad t_i^\alpha \star t_j^\beta = t_j^\beta \star t_i^\alpha \quad \alpha \neq \beta. \quad (5.5)$$

Consider the subalgebra of the enveloping algebra of these relations spanned by polynomials $P(t_i^\alpha)$ that are symmetric under the group S_N permuting different species t_i^α , *i.e.*,

$$P(t_i^\alpha) \in M_N : \quad T_{\alpha\beta} P(t) = P(t) T_{\alpha\beta}, \quad (5.6)$$

where $T_{\alpha\beta}$ are generators of S_N , which exchange species α and β

$$T_{\alpha\beta} = T_{\beta\alpha}, \quad T_{\alpha\beta} t_i^\beta = t_i^\alpha T_{\alpha\beta} \quad (5.7)$$

(no summation over repeated indices). This algebra is isomorphic to $M_N(A)$ via the following identification

$$\begin{aligned} \alpha_i \in M_N(A) : & \quad \sum_{\delta=1}^N t_i^\delta, \\ \alpha_i \alpha_j \in M_N(A) : & \quad \frac{1}{2} \sum_{\delta \neq \beta=1}^N t_i^\delta \star t_j^\beta, \\ \alpha_i \alpha_j \alpha_k \in M_N(A) : & \quad \frac{1}{6} \sum_{\delta \neq \beta \neq \gamma \neq \delta=1}^N t_i^\delta \star t_j^\beta \star t_k^\gamma, \end{aligned} \quad (5.8)$$

etc. Being isomorphic to $M_N(A)$ as a linear space and having proper adjoint action of $l(A)$ generated by t_i , the resulting algebra is isomorphic to $M_N(A)$. For $N = \infty$, this construction gives $M(A)$. Obviously, $M_1(A) = A \oplus \mathbb{K}$.

Algebras $M^N(A)$ result from $M_N(A)$ via factorization of the ideal (3.88). The unit element e_\star of A is realized as $e_\star = \sum_\alpha e_\star^\alpha$. In accordance with Section 3.4, for $M_N(A)$ with finite N , the only possible value of c is N since, for any element f_N of maximal degree N , $e_\star \star F_N = N F_N$. As a result, the naive $N \rightarrow \infty$ limit of $M^N(A)$, via extension of the number of species of t_i^α to infinity may be problematic leading to divergent c . Again, this indicates that algebras $M_c(A)$ may have no direct physical application.

In fact, the difficulties with the extension of HS algebras discussed in Introduction resulted just from the condition that all species of oscillators in (1.4) have common unit element. In other words, the difficulties were due to an attempt to use algebras $M^N(A)$. In the framework of algebras $M_N(A)$ and $M(A)$, oscillators of any sort (index α) have their own unit elements e_\star^α . Single-particle states are realized as $N^{-1/2} \sum_{\alpha=1}^N |\psi_\alpha\rangle$. In particular, the lowest energy single-particle state is represented by

$$N^{-1/2} \sum_{\alpha=1}^N |0_\alpha\rangle, \quad (5.9)$$

where $|0_\alpha\rangle$, represents the lowest energy state in the respective sector. It suffices to require

$$e_\star^\beta |0_\alpha\rangle = \delta_\alpha^\beta |0_\alpha\rangle \quad (5.10)$$

to achieve that the action of generators (1.4) on the state (5.9) remains the same as in the original $N = 1$ module, hence escaping the problem with lowest energies.

The linear space of $M(H_V)$ (5.3) represents the space of multiparticle states of the bulk HS theory which is different from the space of multiparticle states $\sum_{n=0}^\infty \oplus V^n$ of the boundary theory. In field-theoretical terms, algebra $M(H_V)$ and its associated Lie (super)algebras are well

suitable for the description of the bulk multiparticle theory that does not include boundary fields. It may, however, be useful to unify both types of fields in the same framework. For example, such a generalization should underly the extension of the analysis of the AdS_4/CFT_3 HS correspondence of [74] to the case where boundary currents are built from the boundary conformal fields. This can be achieved via the following generalization of the proposed construction.

Consider algebra $H_{V'}$ spanned by elements

$$F \in H_{V'} : \quad F = (f, |v\rangle, \langle v|, \phi) \quad f \in H_V, \quad |v\rangle \in V, \quad \langle v| \in V^*, \quad \phi \in \mathbb{K} \quad (5.11)$$

with the product law

$$F_1 * F_2 = (f_1 \star f_2 + |v_1\rangle\langle v_2|, f_1|v_2\rangle + \phi_2|v_1\rangle, \langle v_1|f_2 + \phi_1\langle v_2|, \phi_1\phi_2 + \langle v_1|v_2\rangle). \quad (5.12)$$

$H_{V'}$ can be interpreted as the algebras of endomorphisms of the space $V' = V \oplus \mathbb{K}$.

Supertrace of H_V generates supertrace of $H_{V'}$

$$str_* F = str_* f + \phi \quad (5.13)$$

provided that the inner product $\langle | \rangle$ is defined via

$$\langle v_1 | v_2 \rangle = (-1)^{\pi_1 \pi_2} str_* (|v_2\rangle \langle v_1|). \quad (5.14)$$

Note that such HS algebras were used in [95] for the description of $2d$ HS gauge theory.

Clearly, $M(H_{V'})$ contains all combinations of multiparticle and multi-antiparticle states of the original boundary theory where V was the space of single-particle states. Since single-particle boundary states can be interpreted as singletons, the resulting construction is analogous to that of singleton strings discussed in [13, 14]. It seems to be most appropriate for the analysis of multiparticle amplitudes in the boundary theory.

Finally, multiparticle algebras admit further extension to brane-like symmetries via algebras $M^p(A)$ defined inductively

$$M^{n+1}(A) = M(M^n(A)) \quad (5.15)$$

with $M^0(A) = A$, $M^1(A) = M(A)$. For the oscillator realization of A by functions of oscillators Y^A , elements of $M^p(A)$ are represented by functions $f(Y)$ of $Y_{i_1 i_2 \dots i_p}^A$ endowed with p copies of indices i_k running from 0 to ∞ , such that $f(Y)$ is symmetric with respect to permutation of $Y_{i_1 i_2 \dots i_p}^A$ for any k . As the variables $Y_{i_1 i_2 \dots i_p}^A$ are reminiscent of the modes on a p -dimensional surface, the algebras $M^p(A)$ are expected to be related to brane-like theories. Note that, in the HS setup, the continuous spectrum difficulty in brane theory discovered in [96] is likely to be resolved in units of the background curvature. The p -brane algebras acting in hypothetical generalized membrane theories, introduced analogously to $m_u(V)$, $m_o(V)$ and $m_{usp}(V)$, we call $m_u^p(V)$, $m_o^p(V)$ and $m_{usp}^p(V)$, respectively.

6 Conclusion

Extensions of HS algebras suggested in this paper are anticipated to underly multiparticle extensions of HS gauge theories containing mixed symmetry fields associated with higher Regge

trajectories of String Theory. An important feature of multiparticle algebras, that differs them from known HS algebras, is that they are not Lie (super)algebras of endomorphisms of unitary modules associated with one or another set of relativistic particles. This modification makes it possible to avoid difficulties of the naive extension of construction of HS algebras to mixed symmetry HS fields.

Known HS algebras $h_u(V)$ can be realized as (matrix valued) Weyl algebra for some set of oscillators Y_A and unit element e_\star or a quotient of some its subalgebra. Multiparticle algebra $m_u(V)$ is realized by symmetric functions $f_n(\tilde{Y}_1, \dots, \tilde{Y}_n)$ of any number $n = 0, 1, 2 \dots$ of variables $\tilde{Y}_\alpha = (Y_{\alpha A}, e_{\star\alpha})$. The spaces F^n of functions $f_n(\tilde{Y}_1, \dots, \tilde{Y}_n)$ with various n are reminiscent of n^{th} higher Regge trajectories in String Theory. A space-time symmetry algebra s belongs to both $h_u(V)$ and $m_u(V)$. Following [2], the idea is to try to formulate a multiparticle HS theory in terms of differential forms $\Omega_n(\tilde{Y}_1, \dots, \tilde{Y}_n)$ of various degrees $p \geq 0$.

Gravitational field is associated with the 1-forms valued in s . The space-time symmetry algebra $s \subset h_u(V)$ can be extended to a larger finite-dimensional subalgebra of $m_u(V)$. For example, in the case where H_V is Weyl algebra, one can consider the algebra spanned by two types of bilinears

$$f_1(Y) = f_{1AB} Y^A \star Y^B, \quad f_2(Y) = f_{2AB} Y^A Y^B, \quad (6.1)$$

where symmetrized tensor product is replaced by usual product. Various $f_1(Y)$ span $sp(2M)$ while $f_2(Y)$ extends it to $sp(2M) \oplus sp(2M)$ where the $sp(2M)$ spanned by $f_1(Y)$ is embedded diagonally. Hence there is more room for the choice of background fields in $m_u(V)$ than in $h_u(V)$.

The problem of increase of vacuum energies mentioned in Introduction does not occur because the $h_u(V)$ -module $V \otimes V$, which describes symmetric massless fields, belongs to the $h_u(V)$ -module \mathcal{V} which is also a $m_u(V)$ -module.

In HS gauge theory, free massless fields are formulated in terms of gauge 1-forms valued in the adjoint representation of the HS algebra and 0-forms C valued in the module often called Weyl module since it contains Weyl tensor in the spin two sector along with its HS generalizations. In the unfolded formulation, all degrees of freedom in the system are represented by 0-forms. Hence, Weyl module treated as a module of the space-time symmetry algebra s is complex equivalent to a unitary module of single-particle states in the system.

Weyl module is realized as the twisted adjoint module with respect to automorphism τ that changes a sign of translations in the AdS algebra, *i.e.*,

$$\tau(L^{ab}) = L^{ab}, \quad \tau(P^a) = -P^a, \quad (6.2)$$

where L^{ab} and P^a are generators of Lorentz transformations and AdS translations, respectively. As explained in Section 3.3, τ induces \mathcal{T} -twisted adjoint $m_u(V)$ -module. Let us call it \mathcal{C} . As a linear space it is isomorphic to the sum of all symmetrized tensor products of C

$$\mathcal{C} = \sum_{n=0}^{\infty} \oplus Sym \underbrace{C \otimes \dots \otimes C}_n. \quad (6.3)$$

As such, it should be complex equivalent to the space of all multiparticle states of the AdS HS theory (not to be confused with the space of multiparticle states of the boundary conformal

theory). This implies that the extension of the HS theory based on $m_u(V)$ should describe all multiparticle states of the original HS theory while $m_u(V)$ is a symmetry that acts on these states. Since C is complex equivalent to a unitary module of the AdS_d algebra s , its symmetrized tensor products and hence \mathcal{C} also do. This suggests that the algebra $m_u(V)$ respects the admissibility condition of [22].

To extend construction to the full nonlinear multiparticle HS theory it is necessary to extend the construction of [2] to various algebras $m_u(V)$ associated with HS algebras $h_u(V)$. Hopefully, solution to this problem can drive us to new understanding of a fundamental theory underlying both String Theory and HS theory.

Another application of the multiparticle algebras is that they are expected to fix unambiguously the form of all correlators of conserved conformal currents of all spins. Indeed, as shown in [72, 75] the form of current operator algebra for conserved conformal HS currents, and hence correlators, is unique for $d > 2$. This fact has been used in [80], where connected parts of n -particle correlators were found with the essential use of their covariance under HS symmetry which determines each of them up to a factor. The multiparticle algebra proposed in this paper relates n -particle correlators with different n . Hence, it should determine all n -particle correlators up to an overall coefficient.

A surprising consequence of the analysis of this paper is that the number of constituent conformal fields \mathcal{N} , is not an essential parameter of the multiparticle algebra. In other words, current operator algebras with different \mathcal{N} are all equivalent, resulting from different basis choices in the same multiparticle algebra. This phenomenon is closely related to the enormous ambiguity in the choice of trace operation in the multiparticle algebra: the same multiparticle algebra can give rise to inequivalent \mathcal{N} -dependent n -point functions once the latter are expressed in terms of different \mathcal{N} -dependent trace operations.

Moreover, beyond a few distinguished bases leading to operator algebra of free currents with different \mathcal{N} , there exist infinitely many bases where the form of current operator algebra and n -point functions do not respect the Wick theorem. This raises an interesting question whether or not this opens a way towards construction of non-free theories. Indeed, once the basis changes within $M(A)$ relate free theories with different \mathcal{N} , which are not equivalent as field theories, more general basis changes, most of which do not respect the Wick theorem, may generate non-free models. (Recall that unfolded equations map operators in the twistor space to conserved space-time currents independently of their construction in terms of free fields; see [35].) The nontrivial part of the story is to check which of the resulting non-linear theories are standard conformal theories in the sense that usual stress tensor (*i.e.*, spin two current) has standard OPE with other primary currents. We hope to consider this interesting issue elsewhere.

Finally, a very interesting problem for the future is to apply the developed technics to $2d$ conformal field theory. At the present stage, this problem is not quite straightforward since the unfolded machinery, which maps the space-time description to the twistor one used in this paper, has not been yet developed far enough for $2d$ conformal models (see, however, [95]).

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